Problem Set #8 Solutions - Physics 210

Problem 1

(a) $L = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2) + e^2/r$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad , \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \quad , \quad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2 \sin^2\theta\, \dot{\phi}$$

$$\Rightarrow H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2\sin^2\theta} - \frac{e^2}{r}$$

(b) $$H = \frac{1}{2m}\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial W}{\partial \theta}\right)^2 + \frac{1}{2mr^2\sin^2\theta}\left(\frac{\partial W}{\partial \phi}\right)^2 - \frac{e^2}{r}$$

Both $p_\phi = L_z$, $p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta} = L^2$ are conserved.

So let $\alpha_\phi = L_z$, $\alpha_\theta^2 = L^2$

If we separate variables so $W = W_r + W_\theta + W_\phi$,

$$\frac{dW_\phi}{d\phi} = \alpha_\phi \quad , \quad \left(\frac{dW_\theta}{d\theta}\right)^2 + \frac{\alpha_\phi^2}{\sin^2\theta} = \alpha_\theta^2 \quad \Rightarrow E = \frac{1}{2m}\left(\left(\frac{dW_r}{dr}\right)^2 + \frac{\alpha_\theta^2}{r^2}\right) - \frac{e^2}{r^2}$$

$$\Rightarrow \left(\frac{dW_r}{dr}\right)^2 + \frac{\alpha_\theta^2}{r^2} = 2m\left(E + \frac{e^2}{r}\right)$$

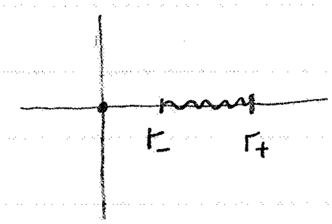
(c) $$J_\phi = \frac{1}{2\pi}\oint \alpha_\phi \, d\phi$$

$$J_\theta = \frac{1}{2\pi}\oint \sqrt{\alpha_\theta^2 - \frac{\alpha_\phi^2}{\sin^2\theta}} \, d\theta$$

$$J_r = \frac{1}{2\pi}\oint \sqrt{2m\left(E + \frac{e^2}{r}\right) - \frac{\alpha_\theta^2}{r^2}} \, dr$$

(d) $$\boxed{J_\phi = \alpha_\phi}$$

$$J_r = \frac{1}{2\pi i}\oint \sqrt{\frac{\alpha_\theta^2}{r^2} - 2m\left(E + \frac{e^2}{r}\right)} \, dr$$, which has a pole at 0



$$\Rightarrow J_r = \frac{1}{2\pi i}\Big(\underbrace{2\pi i \sqrt{\frac{-me^4}{2E}}}_{\text{contour at } \infty} - \underbrace{2\pi i \alpha_\theta}_{\text{residue at } 0} \Big)$$

$$\therefore \boxed{J_r = i\sqrt{\frac{me^4}{2E}} - \alpha_\theta}$$

$$J_\theta = \frac{1}{2\pi i}\oint \sqrt{\frac{\alpha_\phi^2}{\sin^2\theta} - \alpha_\theta^2} \, d\theta$$ $\Leftarrow$ residue at 0 is α_ϕ

$$= \frac{1}{2\pi i}\left(\int_0^{2\pi} \sqrt{\frac{\alpha_\phi^2}{\sin^2\theta} - \alpha_\theta^2} \, d\theta - 2\pi i \alpha_\phi \right)$$

Changing variables: $e^{i\theta} = z \Rightarrow \frac{dz}{d\theta} = iz$

$$\Rightarrow J_\theta = \frac{1}{2\pi i}\left(\oint_{|z|=1} \frac{1}{iz}\sqrt{\frac{-4\alpha_\phi^2}{\left(z-\frac{1}{z}\right)^2} - \alpha_\theta^2}\, dz - 2\pi i \alpha_\phi\right)$$

$$= \frac{1}{2\pi i}\left(\oint \sqrt{\frac{4\alpha_\phi^2}{(z^2-1)^2} + \frac{\alpha_\theta^2}{z^2}}\, dz - 2\pi i \alpha_\phi\right)$$

$$= \frac{1}{2\pi i}\left(2\pi i \alpha_\theta - 2\pi i \alpha_\phi\right) \Rightarrow \boxed{J_\theta = \alpha_\theta - \alpha_\phi}$$

Thus $\quad J_r + J_\theta + J_\phi = i\sqrt{\frac{me^4}{2E}}$

$$\Rightarrow \boxed{E = \frac{-me^4}{2(J_r + J_\phi + J_\theta)^2}}$$

(e) L_z and L^2 are still conserved, so

$$\frac{dW_\phi}{d\phi} = \alpha_\phi, \quad \left(\frac{dW_\theta}{d\theta}\right)^2 + \frac{\alpha_\phi^2}{\sin^2\theta} = \alpha_\theta^2$$

$$\Rightarrow \boxed{J_\phi = \alpha_\phi}, \quad \boxed{J_\theta = \alpha_\theta - \alpha_\phi}$$

$$E = \sqrt{\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right)c^2 + (mc^2)^2} - \frac{e^2}{r}$$

$$= \sqrt{\left(\left(\frac{dW_r}{dr}\right)^2 + \frac{\alpha_\theta^2}{r^2}\right)c^2 + m^2c^4} - \frac{e^2}{r}$$

$$\Rightarrow \left(\frac{dW_r}{dr}\right)^2 = \frac{1}{c^2}\left(E + \frac{e^2}{r}\right)^2 - \frac{\alpha_\theta^2}{r^2} - m^2c^2$$

$$\Rightarrow J_r = \frac{1}{2\pi i}\oint \sqrt{\frac{1}{r^2}\left(\alpha_\theta^2 - \frac{e^4}{c^2}\right) - \frac{1}{r}\frac{2Ee^2}{c^2} + m^2c^2 - \frac{E^2}{c^2}}\, dr$$

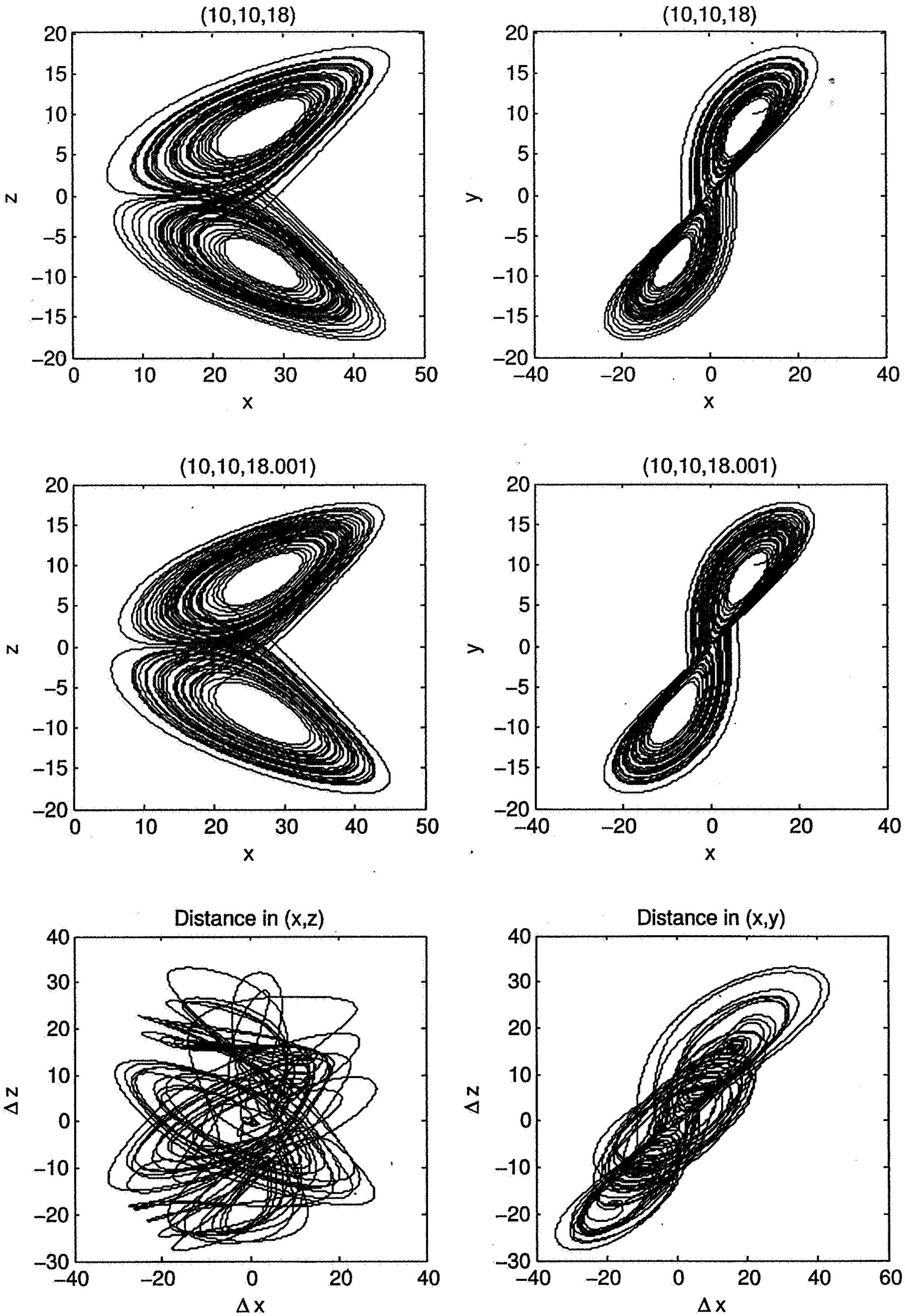
$$= \frac{\sqrt{2Ee^2/c^2}}{2(m^2c^2 - E^2/c^2)} - \sqrt{\alpha_\theta^2 - \frac{e^4}{c^2}} = \frac{1}{c}\sqrt{\frac{e^4}{m^2c^4/E^2 - 1}} - \sqrt{\alpha_\theta^2 - \frac{e^4}{c^2}}$$

So,

$$cJ_r + \sqrt{(J_\theta + J_\phi)^2c^2 - e^4} = \sqrt{\frac{e^4}{m^2c^4/E^2 - 1}}$$

$$\Rightarrow \boxed{\frac{E}{mc^2} = \left(1 + \frac{e^4}{\left(cJ_r + \sqrt{(J_\theta + J_\phi)^2c^2 - e^4}\right)^2}\right)^{-1/2}}$$

PROBLEM 2



(10,10,18)
z
x
(10,10,18)
y
x
(10,10,18.001)
z
x
(10,10,18.001)
y
x
Distance in (x,z)
Δz
Δx
Distance in (x,y)
Δz
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