PROBLEM SET 7 SOLUTIONS - PHYSICS 210

PROBLEM 1

(a) $Q = \ln\left(\frac{\sin p}{q}\right) \qquad P = q\cot p$

$$\mathcal{J} = \begin{pmatrix} \partial Q/\partial q & \partial Q/\partial p \\ \partial P/\partial q & \partial P/\partial p \end{pmatrix} = \begin{pmatrix} -1/q & \cot p \\ \cot p & -1/\sin^2 p \end{pmatrix}$$

By direct verification,

$$\begin{pmatrix} -1/q & \cot p \\ \cot p & -1/\sin^2 p \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -1/q & \cot p \\ \cot p & -1/\sin^2 p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\Rightarrow \quad \mathcal{J}^{+} E \mathcal{J}^{+} = E$$

(b) $Q = \ln(1 + \sqrt{q}\cos p)$, $P = 2(1 + \sqrt{q}\cos p)\sqrt{q}\sin p$

$$\mathcal{J} = \begin{pmatrix} e^{-Q}\cos p/2\sqrt{q} & -e^{-Q}\sqrt{q}\sin p \\ \sin p/\sqrt{q} + \sin 2p & 2\sqrt{q}\cos p + 2q\cos 2p \end{pmatrix}$$

Again, by direct verification, $\mathcal{J} E \mathcal{J}^{T} = E$

(c) $F_3(Q,p) = -(e^Q-1)^2 \tan p$

$$P = -\frac{\partial F_3}{\partial Q} = 2(e^Q-1)\, e^Q \tan p$$

$$q = -\frac{\partial F_3}{\partial p} = (e^Q-1)^2 \frac{1}{\cos^2 p}$$

$$\Rightarrow \quad (e^Q-1)^2 = q\cos^2 P \quad \Rightarrow \quad e^Q = \sqrt{q}\cos p + 1$$

$$\Rightarrow \boxed{Q = \ln(1+\sqrt{q}\cos p)}$$

$$P = 2(\sqrt{q}\cos p)(\sqrt{q}\cos p + 1)\tan p$$

$$\Rightarrow \boxed{P = 2\sqrt{q}\sin p\,(\sqrt{q}\cos p + 1)}$$

(d) $qe^Q = \sin p \quad , \quad P = e^{-Q}\cos p$

Let $\boxed{F(Q,p) = e^{-Q}\cos p}$

Then $q = -\frac{\partial F}{\partial p} = \sin p\, e^{-Q}$ ✓

$P = -\partial F/\partial Q = e^{-Q}\cos p$ ✓

PROBLEM 2

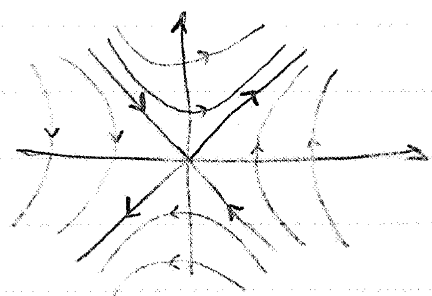
(a) Equilibrium points are $y=0$, $x - Ax^3 = 0$

$\Rightarrow (0,0)$ is equilibrium point for all A

$\Rightarrow (\pm \frac{1}{A}, 0)$ is equilibrium for $A>0$

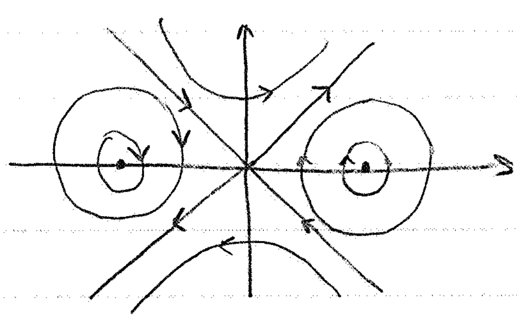
At $(0,0)$, $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \approx \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{cases} \lambda_1 = 1 \text{ for } (1,1) \\ \lambda_2 = -1 \text{ for } (1,-1) \end{cases}$

So for $A<0$:



For $A>0$, at $(\pm \frac{1}{A}, 0)$, $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \approx \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{cases} \lambda_1 = i\sqrt{2} \text{ for } (1, i\sqrt{2}) \\ \lambda_2 = -i\sqrt{2} \text{ for } (1, -i\sqrt{2}) \end{cases}$

So for $A>0$

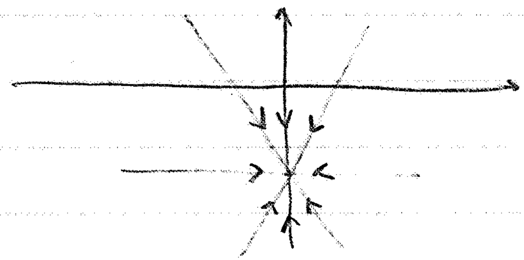


(b) Equilibrium points : $(0, A)$ $\forall A$

$(\pm\sqrt{A}, 0)$ for $A > 0$

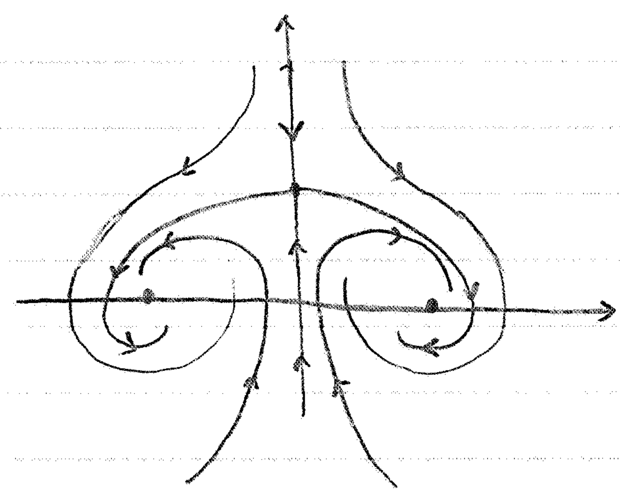
At $(0, A)$, $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} \approx \begin{pmatrix} A & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix} \Rightarrow \begin{cases} \lambda_1 = A & \text{for } (1, 0) \\ \lambda_2 = -1 & \text{for } (0, 1) \end{cases}$

So for $A < 0$



At $(\pm\sqrt{A}, 0)$, $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} \approx \begin{pmatrix} 0 & \pm\sqrt{A} \\ \mp 2\sqrt{A} & -1 \end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix} \Rightarrow \begin{cases} \lambda_1 = \dfrac{-1 + i\sqrt{7}}{2} & \text{for } \left(1, \mp \dfrac{1 \pm i\sqrt{7}}{2\sqrt{A}}\right) \\ \lambda_2 = \dfrac{-1 - i\sqrt{7}}{2} & \text{for } \left(1, \dfrac{\mp 1 \mp i\sqrt{7}}{2\sqrt{A}}\right) \end{cases}$

So for $A > 0$



(c) The equilibrium points are

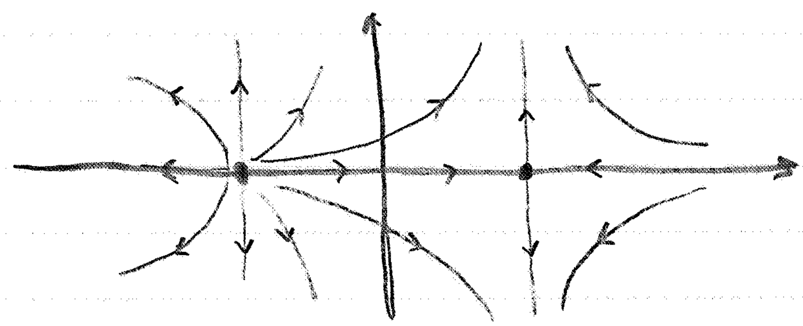
$y = 0, \quad x = 0 \text{ or } x = 1$

$y = 1, \quad x - x^2 - A = 0 \Rightarrow x = \frac{1}{2} \pm \sqrt{\frac{1}{4} - A} \quad \text{for } A < 1/4$

At $(0,0)$, $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \approx \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{cases} \lambda_1 = 1 & \text{for } (1,0) \\ \lambda_2 = 1 & \text{for } (0,1) \end{cases}$

At $(1,0)$, $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \approx \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{cases} \lambda_1 = 1 & \text{for } (0,1) \\ \lambda_2 = -1 & \text{for } (1,0) \end{cases}$

So for $A > 1/4$



At $\left(\frac{1}{2} + \sqrt{\frac{1}{4} - A}, 1\right)$, $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \approx \begin{pmatrix} -2\sqrt{\frac{1}{4} - A} & -2A \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{cases} \lambda_1 = -2\sqrt{\frac{1}{4} - A} & \text{for } (1,0) \\ \lambda_2 = -1 & \text{for } \left(1, \frac{1 - 2\sqrt{\frac{1}{4} - A}}{2A}\right) \end{cases}$

At $\left(\frac{1}{2} - \sqrt{\frac{1}{4} - A}, 1\right)$, $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \approx \begin{pmatrix} 2\sqrt{\frac{1}{4} - A} & -2A \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{cases} \lambda_1 = 2\sqrt{\frac{1}{4} - A} & \text{for } (1,0) \\ \lambda_2 = -1 & \text{for } \left(1, \frac{1 + 2\sqrt{\frac{1}{4} - A}}{2A}\right) \end{cases}$

So for $A < 1/4$