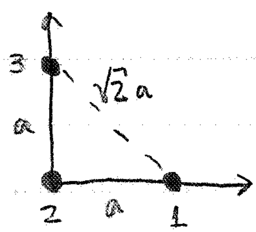
Problem Set #6 Solutions - Physics 210

Problem 1



The equations of motion are

$$
\begin{cases}
\ddot{x}_1 = -\dfrac{K}{m}(x_1 - x_2 - a) - \dfrac{\bar{K}}{\sqrt{2}m}\left(\sqrt{(x_1-x_3)^2 + (y_3-y_1)^2} - \sqrt{2}\,a\right) \\
\ddot{y}_1 = \dfrac{\bar{K}}{\sqrt{2}\,m}\left(\sqrt{(x_1-x_3)^2 + (y_3-y_1)^2} - \sqrt{2}\,a\right)
\end{cases}
$$

$$
\left\{\ddot{x}_2 = \frac{K}{m}(x_1 - x_2 - a) \quad , \quad \ddot{y}_2 = \frac{K}{m}(y_3 - y_2 - a)\right.
$$

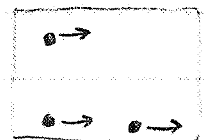
$$
\begin{cases}
\ddot{x}_3 = \dfrac{\bar{K}}{m\sqrt{2}}\left(\sqrt{(x_1-x_3)^2 + (y_3-y_1)^2} - \sqrt{2}\,a\right) \\
\ddot{y}_3 = -\dfrac{K}{m}(y_3 - y_2 - a) - \dfrac{\bar{K}}{\sqrt{2}m}\left(\sqrt{(x_1-x_3)^2 + (y_3-y_1)^2} - \sqrt{2}\,a\right)
\end{cases}
$$

Taylor expanding, we have

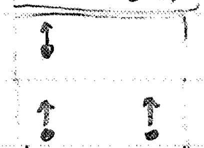
$$
\begin{bmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{x}_2 \\ \ddot{y}_2 \\ \ddot{x}_3 \\ \ddot{y}_3 \end{bmatrix}
= \frac{1}{m}
\begin{bmatrix}
-K-\bar{K}/2 & \bar{K}/2 & K & 0 & \bar{K}/2 & -\bar{K}/2 \\
\bar{K}/2 & -\bar{K}/2 & 0 & 0 & -\bar{K}/2 & \bar{K}/2 \\
K & 0 & -K & 0 & 0 & 0 \\
0 & 0 & 0 & -K & 0 & K \\
\bar{K}/2 & -\bar{K}/2 & 0 & 0 & -\bar{K}/2 & \bar{K}/2 \\
-\bar{K}/2 & \bar{K}/2 & 0 & K & \bar{K}/2 & -K-\bar{K}/2
\end{bmatrix}
\begin{bmatrix} x_1 - a \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 - a \end{bmatrix}
$$

The three eigenmodes with zero eigenvalues correspond to two translations and one rotation:

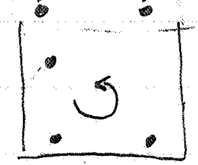
* $\lambda_1 = 0$, $\vec{e}_1 = (1, 0, 1, 0, 1, 0)$



* $\lambda_2 = 0$, $\vec{e}_2 = (0, 1, 0, 1, 0, 1)$

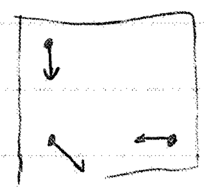


* $\lambda_3 = 0$, $\vec{e}_3 = (1, 2, 1, -1, -2, -1)$

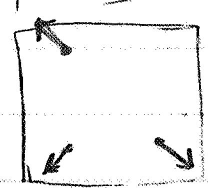


The other 3 eigenvalues / eigenmodes are:

* $\lambda_4 = \frac{2K}{m}$, $\vec{e}_4 = (-1, 0, 1, -1, 0, 1)$

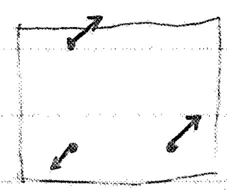


* $\lambda_5 = \frac{-(K+\bar{K}) - \sqrt{\bar{K}^2 - K\bar{K} + K^2}}{m}$



$\vec{e}_5 = (1, s_5, t_5, t_5, s_5, 1)$ with $\begin{cases} s_5 = \left(K - \sqrt{K^2 + \bar{K}^2 - K\bar{K}}\right) / (\bar{K} - K) \\ t_5 = \left(-\bar{K} + \sqrt{K^2 + \bar{K}^2 - K\bar{K}}\right) / (\bar{K} - K) \end{cases}$

* $\lambda_6 = \frac{-(K+\bar{K}) + \sqrt{\bar{K}^2 + K^2 - K\bar{K}}}{m}$



$\vec{e}_6 = (1, s_6, t_6, t_6, s_6, 1)$ with $\begin{cases} s_6 = \left(K + \sqrt{K^2 + \bar{K}^2 - K\bar{K}}\right) / (\bar{K} - K) \\ t_6 = \left(-\bar{K} - \sqrt{K^2 + \bar{K}^2 - K\bar{K}}\right) / (\bar{K} - K) \end{cases}$

PROBLEM 2

a) Let $\vec{r}_{TOT} = \vec{r}_s + \vec{r}$

$$\frac{\vec{F}_{CENT}}{m_M} = -\vec{n}'\times(\vec{n}'\times\vec{r}_s) = \vec{r}_s(\vec{n}'\cdot\vec{n}') - \vec{n}'\underbrace{(\vec{n}\cdot\vec{r}_s)}_{=0} = n'^2\,\vec{r}_s$$

$$\frac{\vec{F}_{GRAV}}{m_N} = -\frac{G_N m_s}{r_{TOT}^3}\,\vec{r}_{TOT} \approx -\frac{G_N m_s}{r_s^3}\left(1 - \frac{3\vec{r}\cdot\hat{r}_s}{r_s}\right)(\vec{r}_s + \vec{r}) \approx -n'^2\left(\vec{r}_{TOT} - 3\hat{r}_s\,\vec{r}\cdot\hat{r}_s\right)$$

$$\text{So}\quad \frac{\vec{F}_{TOT}}{m_N} = \frac{\vec{F}_{CENT}}{m_N} + \frac{\vec{F}_{GRAV}}{m_N} = -n'^2\left(\vec{r} - 3\hat{r}_s\,\vec{r}\cdot\hat{r}_s\right)$$

b) $$\vec{R} = \dot{\vec{r}}\times\frac{\vec{L}}{m} - \gamma\hat{r} \quad\Rightarrow\quad \dot{\vec{R}} = \ddot{\vec{r}}\times\frac{\vec{L}}{m} + \dot{\vec{r}}\times\frac{\dot{\vec{L}}}{m} - \gamma\dot{\hat{r}}$$

With $$\gamma\dot{\hat{r}} = \gamma\,\vec{n}\times\hat{r} = -\frac{\gamma}{r^2}\,\hat{r}\times r^2\,\hat{n} = \frac{\vec{F}_G}{m}\times\frac{\vec{L}}{m}$$

$$\Rightarrow\quad \dot{\vec{R}} = \left(\ddot{\vec{r}} - \vec{F}_G/m\right)\times\frac{\vec{L}}{m} + \dot{\vec{r}}\times\frac{\dot{\vec{L}}}{m}$$

$\ddot{\vec{r}} - \vec{F}_G/m$ is the acceleration due to forces other than the Earth's gravity, so $\ddot{\vec{r}} - \vec{F}_G/m = \vec{F}/m$. Also, Earth's gravity exerts no torque, so $\dot{\vec{L}} = \vec{r}\times\vec{F}$. Therefore,

$$\dot{\vec{R}} = \frac{\vec{F}\times\vec{L}}{m} + \dot{\vec{r}}\times\left(\frac{\vec{r}\times\vec{F}}{m}\right) = \frac{\vec{F}}{m}\times(r^2 n\,\hat{z}) + \dot{\vec{r}}\times\left(\vec{r}\times\frac{\vec{F}}{m}\right)$$

$$= -n'^2\left[\left(\vec{r} - 3\hat{r}_s\,\vec{r}\cdot\hat{r}_s\right)\times(r^2 n\,\hat{z}) + \dot{\vec{r}}\times\left(\vec{r}\times\left(\vec{r} - 3\hat{r}_s\,\vec{r}\cdot\hat{r}_s\right)\right)\right]$$

$$= -n'^2 r^3\left[n\left(\hat{r}\times\hat{z} - 3\hat{r}_s\cdot\hat{r}\;\hat{r}_s\times\hat{z}\right) + 3\hat{r}_s\cdot\hat{r}\;\frac{\dot{\vec{r}}}{r}\times(\hat{r}_s\times\hat{r})\right]$$

Let s be the angle from the fixed direction to $\hat{r}_s$, $\begin{cases} \hat{r}_s \cdot \hat{r} = \cos(f-s) \\ \hat{r}_s \times \hat{r} = \sin(f-s)\hat{z} \end{cases}$

$$\dot{\vec{R}} = -n'^2 r^3 \left[n\left(\hat{r}\times\hat{z} - 3\cos(f-s)\,\hat{r}_s\times\hat{z}\right) + 3\sin(f-s)\cos(f-s)\frac{\dot{\vec{r}}}{r}\times\hat{z} \right]$$

$$\Rightarrow \dot{\vec{R}}\cdot\hat{y} = -n'^2 r^3 \left[n\left(\cos f - 3\cos(f-s)\cos s\right) + 3n\sin(f-s)\cos(f-s)\cos f \right]$$

Averaging $\dot{\vec{R}}\cdot\hat{y}$ over solar and lunar periods:

$$\dot{\vec{R}}\cdot\hat{y} = \frac{3}{4} e n'^2 r^3 n. \quad \text{With } \frac{\gamma}{r^2} = n^2 r \Rightarrow \dot{\vec{R}}\cdot\hat{y} = \frac{3}{4}\gamma e \frac{n'^2}{n}$$

The Runge-Lenz vector has magnitude γe and points towards the perigee, so,

$$|\dot{\vec{R}}| = |\vec{R}||\dot{\hat{R}}| = \gamma e \dot{\omega}$$

$$\Rightarrow \frac{\dot{\omega}}{n} = \frac{3}{4}\left(\frac{n'}{n}\right)^2$$

c) $r = \frac{c^2}{\gamma}$ for a circular orbit.

In the Moon's plane, $\vec{r} = \frac{c^2}{\gamma}(\cos f, \sin f, 0)$. Rotating around $\hat{y}$ by $\Omega - \pi/2$ we get the coordinate system of interest:

$$\vec{r} = \frac{c^2}{\gamma}\begin{pmatrix} \cos(\Omega-\pi/2) & 0 & -\sin(\Omega-\pi/2) \\ 0 & 1 & 0 \\ \sin(\Omega-\pi/2) & 0 & \cos(\Omega-\pi/2) \end{pmatrix}\begin{pmatrix} \cos f \\ \sin f \\ 0 \end{pmatrix} \Rightarrow \vec{r} = \frac{c^2}{\gamma}\left(\cos f\sin\Omega, \sin f, -\cos f\cos\Omega\right)$$

$$\dot{\vec{c}} = \frac{\dot{\vec{L}}}{m} = \vec{r} \times \frac{\vec{F}}{m} = 3n'^2 r^2 \hat{r}_s \cdot \hat{r} (\hat{r} \times \hat{r}_s)$$

With $\hat{r}_s = (\cos s, \sin s, 0)$

$$\Rightarrow \dot{\vec{c}} \cdot \hat{y} = 3n'^2 r^2 (\cos f \sin\Omega \cos s + \sin f \sin s)(\cos f \cos\Omega \cos s)$$

Averaging over cycles $\Rightarrow$ $\dot{\vec{c}} \cdot \hat{y} = -\frac{3}{4} n'^2 r^2 \sin\Omega \cos\Omega$

The h-ecliptic component of $\vec{c} = c(\cos\Omega, 0)$, so $|\dot{\vec{c}}| = c\cos\Omega\dot{h}$

$$\Rightarrow \frac{\dot{h}}{n} = -\frac{3}{4}\left(\frac{n'}{n}\right)^2 \sin\Omega \approx \frac{3}{4}\left(\frac{n'}{n}\right)^2$$