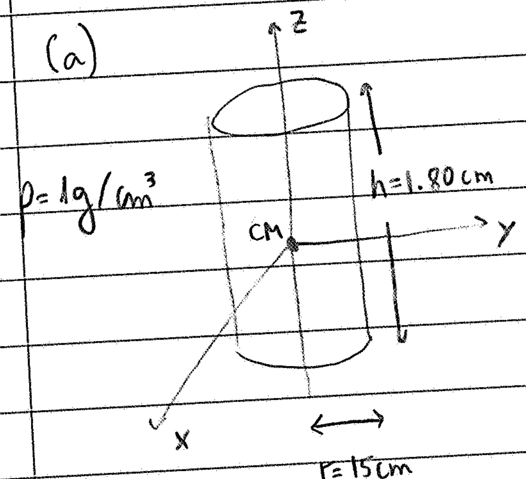
PHYSICS 210 - PROBLEM SET #5

PROBLEM 1

(a)



With the frame orientation in the figure, the inertia tensor is diagonal and $I^{xx} = I^{yy}$.

$$I^{ab} = \int dm \left(r^2 \delta^{ab} - r^a r^b\right)$$

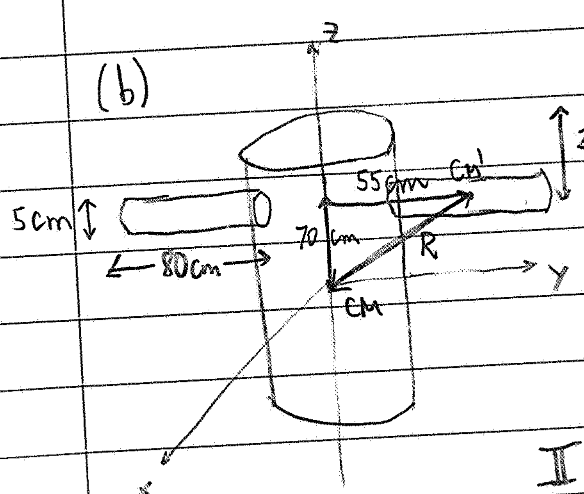
$$= \int dx\,dy\,dz\, \rho \left(r^2 \delta^{ab} - r^a r^b\right)$$

$$I^{xx} = \int dx\,dy\,dz\, \rho\left(y^2 + z^2\right) = \rho\pi h \frac{r^4}{4} + \rho\pi \frac{h^3 r^2}{12} = I^{yy}$$

$$I^{zz} = \int dx\,dy\,dz\, \rho\left(x^2 + y^2\right) = \rho\pi h \frac{r^4}{2}$$

$$\Rightarrow \mathbb{I} = \rho\pi h r^2 \begin{pmatrix} \frac{r^2}{4} + \frac{h^2}{12} & & \\ & \frac{r^2}{4} + \frac{h^2}{12} & \\ & & \frac{r^2}{2} \end{pmatrix} \approx \begin{pmatrix} 35 & & \\ & 35 & \\ & & 1.4 \end{pmatrix} Kg\,m^3$$

(b)



$$R = \sqrt{(70cm)^2 + (55\,cm)^2}$$

The moment of inertia of each arm around their CM is

$$\mathbb{I}^{ARM}_{CM'} = \rho\pi h' r'^2 \begin{pmatrix} r'^2/4 + h'^2/12 & 0 & 0 \\ 0 & r'^2/2 & 0 \\ 0 & 0 & \frac{r'^2}{4} + \frac{h'^2}{12} \end{pmatrix}$$

Using the parallel axis theorem, the moment around the CM of the "body" is

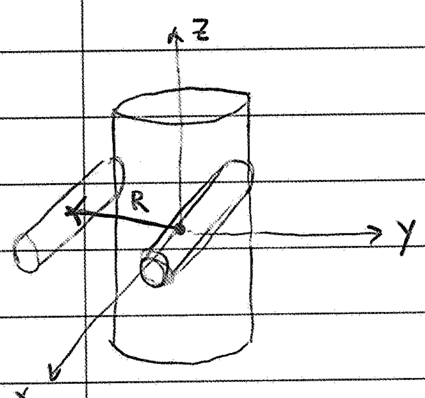
$$I^{ab} = I^{ab}_{BODY} + 2\, I^{ab}_{ARM} + \rho\pi r'^2 h' \left(R^2\delta^{ab} - r_1^a r_1^b\right) + \rho\pi r'^2 h' \left(R^2\delta^{ab} - r_2^a r_2^b\right)$$

$$I^{ab} = I^{ab}_{BODY} + 2\, I^{ab}_{ARM} + \rho\pi r'^2 h' \left[\begin{pmatrix} R^2 & 0 & 0 \\ 0 & R^2-y_1^2 & -y_1 z_1 \\ 0 & -y_1 z_1 & R^2 - z_1^2 \end{pmatrix} + \begin{pmatrix} R^2 & 0 & 0 \\ 0 & R^2-y_2^2 & -y_2 z_2 \\ 0 & -y_2 z_2 & R^2 - z_2^2 \end{pmatrix}\right],$$

with $y_1 = -y_2 = 55\,\text{cm}$ and $z_1 = z_2 = -70\,\text{cm}$

$$\Rightarrow \mathbb{I} = \begin{pmatrix} 45.7 & & \\ & 41.2 & \\ & & 5.9 \end{pmatrix} \text{Kg m}^2$$

c) Same as in (b), but now $y_1 = -y_2 = 20\,\text{cm}$, $x_1 = x_2 = 40\,\text{cm}$, $z_1 = z_2 = -70\,\text{cm}$, $R^2 = x^2 + y^2 + z^2$

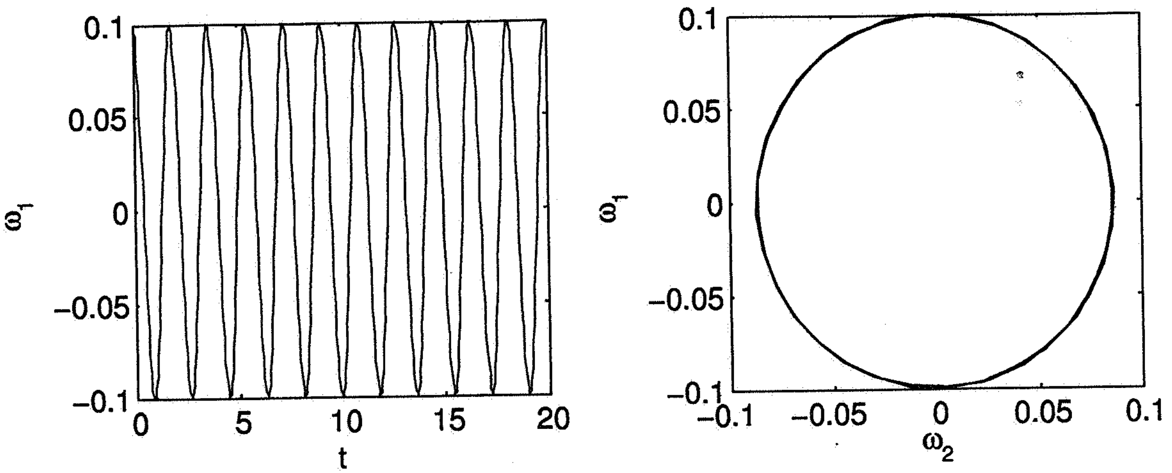


$$I^{ab} = I^{ab}_{BODY} + 2\rho\pi r'^2 h' \begin{pmatrix} \frac{r'^2}{4} + \frac{h'^2}{12} + y^2 + z^2 & 0 & |xz| \\ 0 & \frac{r'^2}{2} + z^2 + x^2 & 0 \\ |xz| & 0 & \frac{r'^2}{4} + \frac{h'^2}{12} + y^2 + x^2 \end{pmatrix}$$

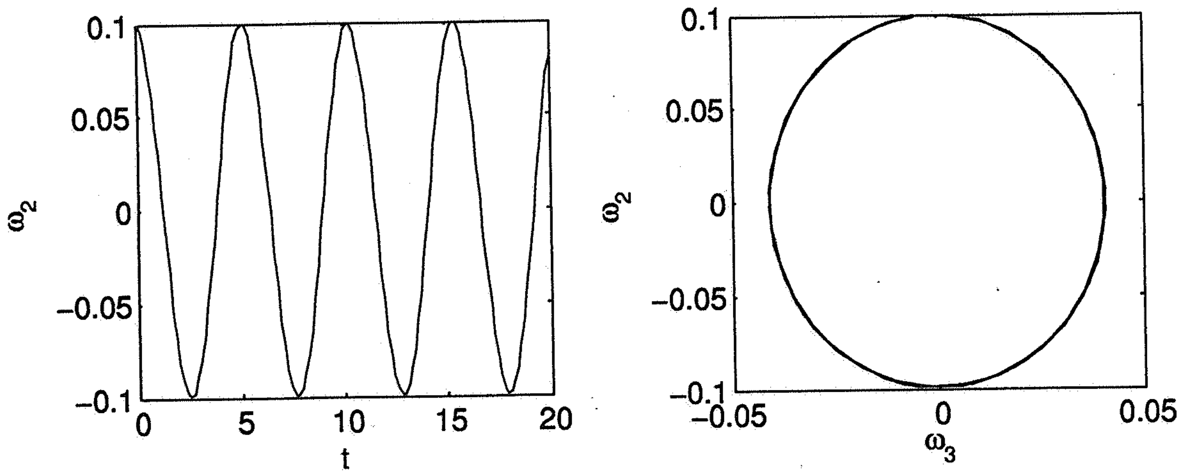
$$\mathbb{I} = \begin{pmatrix} 41.5 & & 3.5 \\ & 43.9 & \\ 3.5 & & 3.8 \end{pmatrix} \text{Kg m}^2$$

PROBLEM 2

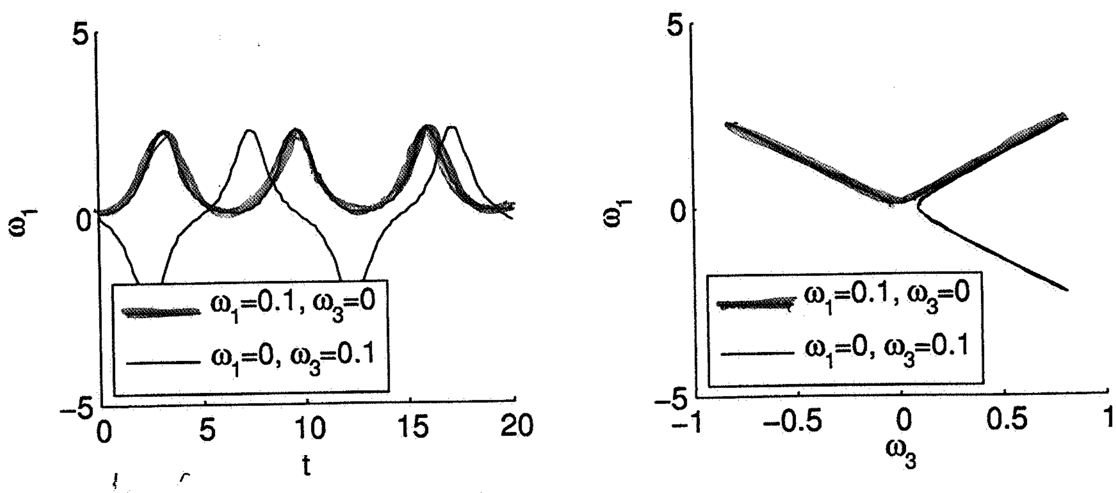
a)



b)



c)



d) $\omega_1^2 - 8\omega_3^2$ $(= 4T - L^2)$ is a conserved quantity, explaining the hyperbolas in the ω_1 vs ω_3 plot. Similarly, ω_1 vs ω_2 and ω_2 vs ω_3 are ellipses.

PROBLEM 3

a) On a moving particle, $\vec{F}_{COR} = -2m\,\vec{\omega}\times\vec{v}$

For a given differential volume of the top,

$$d\vec{\tau} = \vec{r}\times d\vec{F} = \vec{r}\times(-2\,dm\,\vec{\omega}\times\vec{v}) = -2\,dm\,\vec{r}\times(\vec{\omega}\times(\vec{\Omega}\times\vec{r}))$$

Integrating over all differential volumes of the top, with

$dm = \rho(\vec{r})\,d^3\vec{r}$:

$$\vec{\tau} = -2\int d^3\vec{r}\,\rho(\vec{r})\,\vec{r}\times(\vec{\omega}\times(\vec{\Omega}\times\vec{r}))$$

Using the identity

$$\vec{r}\times(\vec{\omega}\times(\vec{\Omega}\times\vec{r})) = \vec{r}\times(\vec{\Omega}(\vec{\omega}\cdot\vec{r}) - \vec{r}\,\underbrace{\cancel{(\vec{\omega}\cdot\vec{\Omega})}}_{=0}) = (\vec{r}\times\vec{\Omega})(\vec{r}\cdot\vec{\omega}),$$

and $\vec{r}\times\vec{\Omega} = \Omega(r_2\hat{\imath} - r_1\hat{\jmath})$

$\vec{r}\cdot\vec{\omega} = r_1\omega_1 + r_2\omega_2 + r_3\omega_3$,

we have

$$\vec{\tau} = -2\Omega\int d^3\vec{r}\,\rho(\vec{r})\,(r_2\hat{\imath} - r_1\hat{\jmath})(r_1\omega_1 + r_2\omega_2 + r_3\omega_3)$$

Because this is a symmetrical top,

$$\int d^3\vec{r}\,\rho(\vec{r})\,r_a r_b = 0 \quad \text{for } a\neq b, \quad \text{and} \quad \int d^3\vec{r}\,\rho(\vec{r})\,r_1^2 = \int d^3\vec{r}\,\rho(\vec{r})\,r_2^2,$$

Hence,

$$\vec{\tau} = -2\Omega \int d^3\vec{r}\, \rho(\vec{r})\, (r_1^2 + r_2^2)\, \frac{(\omega_2 \hat{i} - \omega_1 \hat{j})}{2}$$

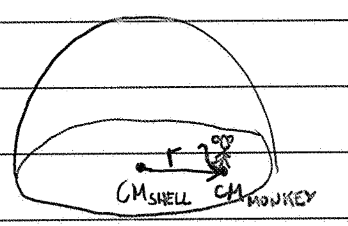
$$\Rightarrow \boxed{\vec{\tau} = -I_3\, \vec{\omega} \times \vec{\Omega}}$$

b) $\vec{\omega} = \vec{\omega}_{\parallel} + \vec{\omega}_{\perp}$, where $\vec{\omega}_{\parallel}$ is horizontal and $\vec{\omega}_{\perp}$ is vertical. The torque component from $\vec{\omega}_{\perp} \times \vec{\Omega}$ is countered by the horizontal constraint of the gyroscope, so

$$\ddot{\phi} = -\frac{I_3 |\vec{\omega}_{\parallel}|\, \Omega \sin\phi}{I_1} = -\frac{I_3}{I_1}\, \Omega\, \omega \cos\theta \sin\phi$$

PROBLEM 4

a)



$$\begin{cases} M\vec{r}^{\,CM}_{SHELL} + \frac{M}{4}\vec{r}^{\,CM}_{MONKEY} = 0 \\ \vec{r}^{\,CM}_{MONKEY} - \vec{r}^{\,CM}_{SHELL} = \vec{r} \end{cases} \Rightarrow \quad \vec{r}^{\,CM}_{MONKEY} = \frac{4\vec{r}}{5}, \quad \vec{r}^{\,CM}_{SHELL} = -\frac{\vec{r}}{5}$$

So the distance of the center of mass of the shell from the total center of mass is $|\vec{r}^{\,CM}_{SHELL}| = r/5$.

b) $$I^{ab}_{CM} = I^{ab}_{SHELL} + M\left(r^2_{SHELL}\delta^{ab} - r^a_{SHELL} r^b_{SHELL}\right) + \frac{M}{4}\left(r^2_{MONKEY}\delta^{ab} - r^a_{MONKEY} r^b_{MONKEY}\right)$$

$$= \begin{pmatrix} Ma^2/4 & & \\ & Ma^2/4 & \\ & & Ma^2/2 \end{pmatrix} + \begin{pmatrix} 0 & & \\ & Mr^2/25 & \\ & & Mr^2/25 \end{pmatrix} + \begin{pmatrix} 0 & & \\ & 4Mr^2/25 & \\ & & 4Mr^2/25 \end{pmatrix}$$

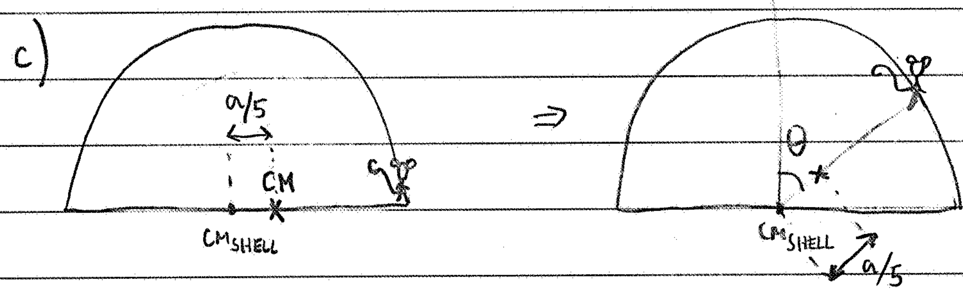
$$I_{CM} = \begin{pmatrix} Ma^2/4 & & \\ & Ma^2/4 + Mr^2/5 & \\ & & Ma^2/2 + Mr^2/5 \end{pmatrix}$$

given that $\vec{L}$ is conserved :

$$\frac{Ma^2}{2}\Omega\hat{z} = M\left(\frac{a^2}{2} + \frac{r^2}{5}\right)\Omega(r)\hat{z}$$

$$\Rightarrow \boxed{\Omega(r) = \frac{\Omega}{1 + \frac{2}{5}\frac{r^2}{a^2}}}$$

c)



Given that the distance between the monkey and the center of mass of the disc is "a" for all "θ", the distance between the center of mass of the ship and the center of mass of the shell is $d(\theta) = a/5$.

d) The center of mass is at $\vec{r} = \left(\frac{a}{5}\sin\theta, 0, \frac{a}{5}\cos\theta\right)$

$$I^{ab}_{CM} = I^{ab}_{CM-SHELL} + M\left(r^2_{SHELL}\delta^{ab} - r^a_{SHELL}r^b_{SHELL}\right) + \frac{M}{4}\left(r^2_{MONKEY}\delta^{ab} - r^a_{MONKEY}r^b_{MONKEY}\right)$$

$$= I^{ab}_{CM-SHELL} + \begin{pmatrix} \frac{Ma^2\cos^2\theta}{25} & 0 & -\frac{Ma^2\sin 2\theta}{50} \\ 0 & \frac{Ma^2}{25} & 0 \\ -\frac{Ma^2\sin 2\theta}{50} & 0 & \frac{Ma^2\sin^2\theta}{25} \end{pmatrix} + \begin{pmatrix} \frac{4Ma^2\cos^2\theta}{25} & 0 & -\frac{4Ma^2\sin 2\theta}{50} \\ 0 & \frac{4Ma^2}{25} & 0 \\ -\frac{4Ma^2\sin 2\theta}{50} & 0 & \frac{4Ma^2\sin^2\theta}{25} \end{pmatrix}$$

$$\Rightarrow I_{CM} = Ma^2 \begin{pmatrix} \frac{1}{4} + \frac{\cos^2\theta}{5} & 0 & -\frac{\sin 2\theta}{10} \\ 0 & 9/20 & 0 \\ -\frac{\sin 2\theta}{10} & 0 & \frac{1}{2} + \frac{\sin^2\theta}{5} \end{pmatrix}$$

e) The inverse of I_{CM} is

$$I_{CM}^{-1} = \frac{1}{Ma^2} \begin{pmatrix} -4 + \frac{56}{8+\cos 2\theta} & 0 & \frac{4\sin 2\theta}{8+\cos 2\theta} \\ 0 & 20/9 & 0 \\ \frac{4\sin 2\theta}{8+\cos 2\theta} & 0 & 4 - \frac{18}{8+\cos 2\theta} \end{pmatrix}$$

Now let $\vec{L} = (\sin\eta\cos\phi, \sin\eta\sin\phi, \cos\eta)$ in the frame fixed on the shell.

So,

$$E(\eta,\phi) = \frac{1}{2} L_i(\eta,\phi) \, (I_{CM}^{-1})_{ij} \, L_j(\eta,\phi)$$

For a given θ, one has to find η an ϕ that minimizes $E(\eta,\phi)$. That gives the ship orientation w.r.t. θ.

When the monkey reaches the top, i.e., for $\theta = 0$,

$$E(\eta,\phi)\Big|_{\theta=0} = \frac{1 + \sin^2\eta}{9}$$

which is minimized for $\eta = 0$, i.e., the ship rotating about the original $\hat{z}$ axis.