PHYSICS 210 - PROBLEM SET 4 SOLUTIONS

PROBLEM 1

(a) Write $\vec{A}$ as $A_K = -\frac{1}{2}\epsilon_{Kmn} r_m B_n$

Then $\vec{B} = \vec{\nabla}\times\vec{A} \Rightarrow B_i = \epsilon_{ijK}\partial_j A_K = -\frac{1}{2}\epsilon_{ijK}\epsilon_{Kmn}\partial_j(r_m B_n)$

$$= -\frac{1}{2}\epsilon_{ijK}\epsilon_{Kmn}\delta_{jm}B_n = \frac{1}{2}2\delta_{in}B_n = B_i$$

(b) The Lagrangian is $L = \frac{1}{2}\sum_i\left(m_i\dot{\vec{r}}_i^2 - \frac{q}{c}\dot{\vec{r}}_i\cdot\vec{A}_i\right)$

From which one obtains the equations of motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\vec{r}}_i}\right) = \frac{\partial L}{\partial\vec{r}_i} \Rightarrow \frac{d}{dt}\left(m\dot{\vec{r}}_i - \frac{q}{c}\vec{A}_i\right) = -\frac{q}{c}\vec{\nabla}\cdot(\dot{\vec{r}}_i\cdot\vec{A}_i)$$

$$= -\frac{q}{c}\left[\dot{\vec{r}}_i\times(\vec{\nabla}\times\vec{A}_i) + (\dot{\vec{r}}_i\cdot\vec{\nabla})\vec{A}_i\right]$$

$$\Rightarrow m_i\ddot{\vec{r}}_i - \frac{q}{c}\frac{d\vec{A}_i}{dt} = -\frac{q}{c}\dot{\vec{r}}_i\times(\vec{\nabla}\times\vec{A}_i) - \frac{q}{c}\frac{d\vec{A}_i}{dt}$$

$$\Rightarrow m_i\ddot{\vec{r}}_i = -\frac{q}{c}\dot{\vec{r}}_i\times\vec{B} = 2m\dot{\vec{r}}_i\times\left(-\frac{q}{2mc}\vec{B}\right) = 2m\dot{\vec{r}}_i\times\vec{\Omega}_L$$

$\Rightarrow \boxed{\vec{F} = 2\dot{\vec{r}}_i\times\vec{\Omega}_L}$, which is the Coriollis' force, and therefore the effects of $\vec{B}$ can be replaced by a rotating coordinate system with angular velocity $\vec{\Omega}_L$ (the centrifugal force is 2nd order in $\vec{\Omega}_L$ and therefore negligible to this approximation).

PROBLEM 2

(a) $\chi(\alpha) = \mathrm{tr}[R_{\hat{n}}(\alpha)] = \sum_i R_{\hat{n}}(\alpha)_{ii}$

Since trace is basis independent, we can evaluate it in a basis where $\hat{n} = \hat{z}$, so that

$$\mathbb{R}_{\hat{z}}(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \boxed{\chi(\alpha) = 1 + 2\cos\alpha}$$

(b) $$\mathrm{Tr}[R_{\hat{n}}(\alpha)] = \mathrm{Tr}[R_{\hat{z}}(\phi) R_{\hat{y}}(\theta) R_{\hat{z}}(\psi)]$$
$$= \mathrm{Tr}[R_{\hat{z}}(\psi) R_{\hat{z}}(\phi) R_{\hat{y}}(\theta)]$$
$$= \mathrm{Tr}[R_{\hat{z}}(\psi+\phi) R_{\hat{y}}(\theta)]$$
$$= \mathrm{Tr}\left[\begin{pmatrix} \cos(\psi+\phi) & \sin(\psi+\phi) & 0 \\ -\sin(\psi+\phi) & \cos(\psi+\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}\right]$$
$$= \mathrm{Tr}\left[\begin{pmatrix} \cos(\psi+\phi)\cos\theta & xxx & xxx \\ xxx & \cos(\psi+\phi) & xxx \\ xxx & xxx & \cos\theta \end{pmatrix}\right]$$
$$= \cos(\psi+\phi)(1+\cos\theta) + \cos\theta$$

$$\Rightarrow \quad 1 + 2\cos\alpha = \cos(\psi+\phi)(1+\cos\theta) + \cos\theta$$

$$4\cos^2\frac{\alpha}{2} - 1 = 2\cos^2\frac{\theta}{2}\cos(\psi+\phi) + 2\cos^2\frac{\theta}{2} - 1$$

$$4\cos^2\frac{\alpha}{2} = 2\cos^2\frac{\theta}{2}\left(\cos(\psi+\phi) + 1\right)$$

$$\Rightarrow \boxed{\cos^2\frac{\alpha}{2} = \cos^2\frac{\theta}{2}\cos\frac{\psi+\phi}{2}}$$

PROBLEM 3

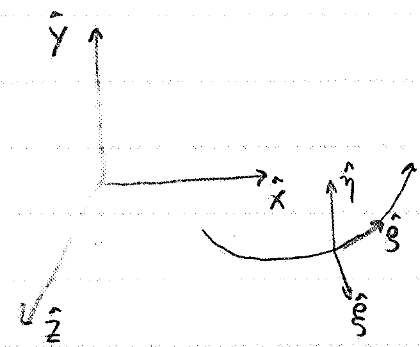
(a) $\hat{\zeta} = R\hat{z}$

$\hat{\zeta} = R_z(\phi)\, R_y(\theta)\, R_z(\psi)\, \hat{z}$

$$\hat{\zeta} = R_z(\psi)\, R_y(\theta)\, \hat{z} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\phi \sin\theta \\ \sin\phi \sin\theta \\ \cos\theta \end{pmatrix}$$

$$\Rightarrow \vec{V} = v\hat{\zeta} = v(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$$

(b) For a banked turn



$\phi = 0$ since the jet trajectory is on the horizontal plane

$\dot{\theta} = \omega$ describing the change in direction

$\psi =$ const since the plane banks

So $\vec{V} = v(\sin\theta, 0, \cos\theta)$ and $\vec{x} = \int dt\, \vec{v} = \int dt\, v(\sin\theta(t), 0, \cos\theta(t))$

given that the initial velocity is in the $\hat{z}$ direction,

$\dot{\theta} = \omega \Rightarrow \theta(t) = \omega t$, so,

$$\vec{x}(t) = \int dt\, v(\sin\omega t, 0, \cos\omega t) = \frac{v}{\omega}(-\cos\omega t, 0, \sin\omega t)$$

(c) In this case $\vec{v} = (\sin\omega t \cos\phi,\ \sin\omega t \sin\phi,\ \cos\omega t)$ and

$$\vec{x}(t) = \int dt\, \vec{v} = \frac{v}{\omega}\left(-\cos\omega t \cos\phi,\ -\cos\omega t \sin\phi,\ \sin\omega t\right)$$

(d)
$$\vec{\Omega} = a\hat{\zeta} + b\hat{\eta} = [\dot{\phi}\cos\theta + \dot{\psi}]\hat{\zeta} + [\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi]\hat{\eta} + [-\dot{\phi}\sin\theta\cos\psi + \dot{\theta}\sin\psi]\hat{\xi}$$

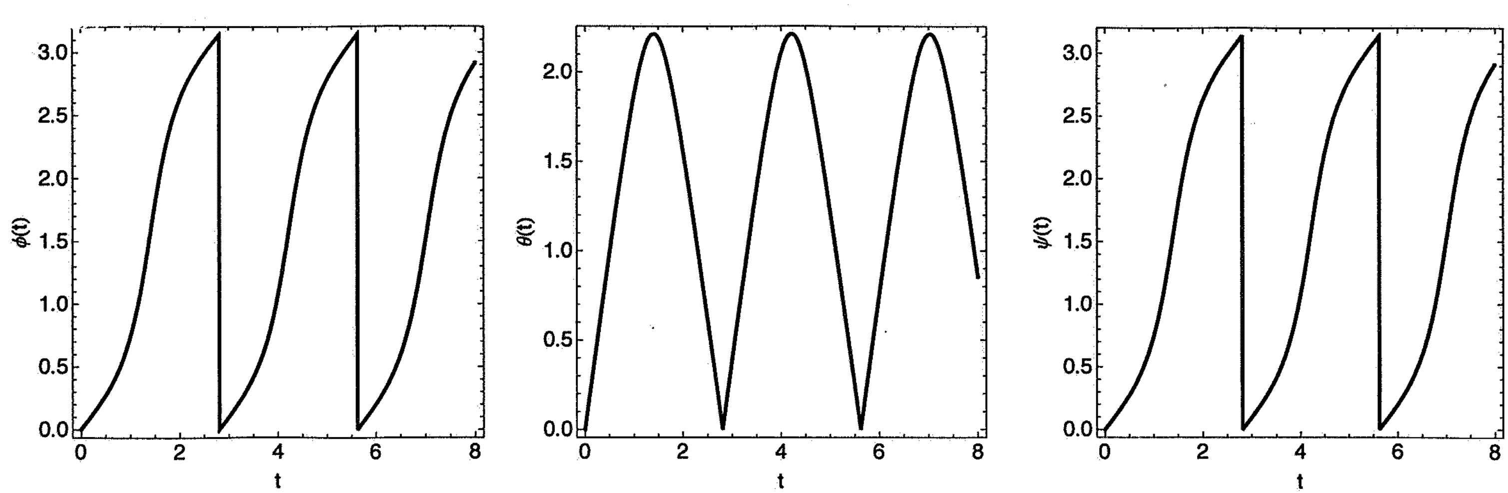
$$\Rightarrow \quad a = \dot{\phi}\cos\theta + \dot{\psi}, \quad b = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi, \quad 0 = -\dot{\phi}\sin\theta\cos\psi + \dot{\theta}\sin\psi$$

Solving for $\dot{\theta}, \dot{\phi}, \dot{\psi}$ $\Rightarrow$
$$\dot{\phi} = b\frac{\sin\psi}{\sin\theta}, \quad \dot{\theta} = b\cos\psi, \quad \dot{\psi} = a - b\frac{\sin\psi}{\tan\theta}$$

(e) To first order in θ, ϕ and ψ

$$\dot{\phi} = \frac{\psi}{\theta}b, \quad \dot{\theta} = b, \quad \dot{\psi} = a - \frac{\psi}{\theta}b \quad \Rightarrow \quad \theta = bt, \quad \phi = \frac{at}{2}, \quad \psi = \frac{at}{2}$$

(f) The numerical solution of eqs. obtained in part (d) with $a = 1\,s^{-1}$, $b = 2\,s^{-1}$ and initial conditions $\theta(\epsilon) = b\epsilon$, $\phi(\epsilon) = a\epsilon/2$ and $\psi(\epsilon) = a\epsilon/2$, with $\epsilon = 0.001\,s$ are:



(g) given that $v_y = V \sin\theta(t) \sin\phi(t)$,

$y(t) = \int dt \, (500\, m/s) \sin\theta(t) \sin\phi(t)$, which is plotted below: