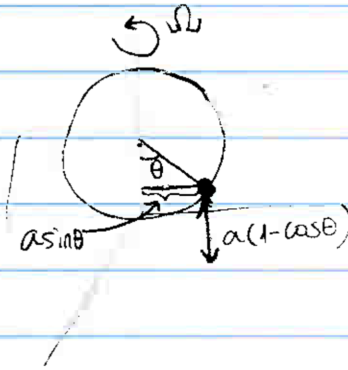
Physics 210 - Problem Set #3 - Solutions

Problem 1



a) $$\boxed{L = \frac{1}{2} m (a\dot{\theta})^2 + \frac{1}{2} m (a\sin\theta\, \Omega)^2 - mga(1-\cos\theta)}$$

b) The equations of motion are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta} \Rightarrow \frac{d}{dt}\left(ma^2\dot{\theta}\right) = ma^2\Omega^2 \sin\theta\cos\theta - mga\sin\theta$$

The condition for circular motion $\theta = \theta_0$ implies that $\dot{\theta} = \ddot{\theta} = 0$

$$\Rightarrow ma^2\Omega^2 \sin\theta_0 \cos\theta_0 - mga\sin\theta_0 = 0 \Rightarrow \boxed{\cos\theta_0 = \frac{g}{a\Omega^2}}$$

c) Substituting $\theta \rightarrow \theta_0 + \delta\theta$ on the eq. of motion $\ddot{\theta} = \Omega^2 \sin\theta\cos\theta - \frac{g}{a}\sin\theta$

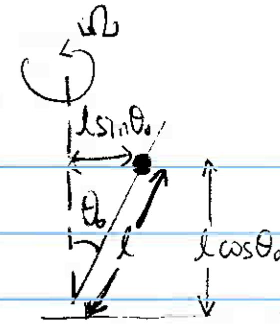
we have: $$\delta\ddot{\theta} = \Omega^2 \sin(\theta_0 + \delta\theta)\cos(\theta_0 + \delta\theta) - \frac{g}{a}\sin(\theta_0 + \delta\theta)$$

$$\Rightarrow \delta\ddot{\theta} = \Omega^2 (\sin\theta_0 + \cos\theta_0\,\delta\theta)(\cos\theta_0 - \sin\theta_0\,\delta\theta) - \frac{g}{a}(\sin\theta_0 + \cos\theta_0\,\delta\theta)$$

$$\Rightarrow \delta\ddot{\theta} = \underbrace{\Omega^2 \sin\theta_0\cos\theta_0 - \frac{g}{a}\sin\theta_0}_{=0} - \Omega^2(-\sin^2\theta_0 + \cos^2\theta_0)\,\delta\theta - \frac{g}{a}\cos\theta_0\,\delta\theta$$

$$\Rightarrow \delta\ddot{\theta} = -\left(\frac{g^2}{a^2\Omega^2} - \Omega^2\right)\delta\theta \Rightarrow \boxed{\omega = \sqrt{\frac{g^2}{a^2\Omega^2} - \Omega^2}}$$

d) In this case the Lagrangian and equations of motion are:



$$L = \frac{1}{2} m \dot{l}^2 + \frac{1}{2} m (l \sin\theta_0 \Omega)^2 - mgl\cos\theta_0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{l}}\right) = \frac{\partial L}{\partial l} \Rightarrow \qquad \ddot{l} = l\sin^2\theta_0\,\Omega^2 - g\cos\theta_0$$

The condition for circular motion, $\ddot{l} = 0$, implies:

$$l_0 \sin^2\theta_0\,\Omega^2 - g\cos\theta_0 = 0 \Rightarrow \boxed{l_0 = \frac{g\cos\theta_0}{(\sin\theta_0\,\Omega)^2}}$$

For small displacements, $l = l_0 + \delta l$:

$$\delta\ddot{l} = \cancel{l_0 \sin^2\theta_0\,\Omega^2 - g\cos\theta_0}_{=0} + \delta l \sin^2\theta_0\,\Omega^2$$

$$\Rightarrow \delta\ddot{l} = \underbrace{\sin^2\theta_0\,\Omega^2}_{>0}\,\delta l$$

$>0 \Rightarrow$ the orbit is unstable

PROBLEM 2

a) We have $L = \frac{1}{2} m \dot{r}_i \dot{r}_i + \frac{mg}{\sqrt{r_i r_i}}$ and

$$\Delta r_i = 2 \dot{r}_i r_j \alpha_j - r_j \dot{r}_j \alpha_i - r_i \dot{r}_j \alpha_j$$

So $L + \delta L = \frac{1}{2} m (\dot{r}_i + \Delta \dot{r}_i)(\dot{r}_i + \Delta \dot{r}_i) + \frac{mg}{\sqrt{(r_i + \Delta r_i)(r_i + \Delta r_i)}}$

$$L + \delta L = \frac{1}{2} m \dot{r}_i \dot{r}_i + m \dot{r}_i \Delta \dot{r}_i + \frac{mg}{r}\left(1 - \frac{r_i \Delta r_i}{r^2}\right)$$

$$L + \delta L = L + m \dot{r}_i \Delta \dot{r}_i - \frac{mg}{r^3} r_i \Delta r_i$$

$$\delta L = m \frac{d}{dt}(\dot{r}_i \Delta r_i) - m \ddot{r}_i \Delta r_i - \frac{mg}{r^3} r_i \Delta r_i$$

$$= m \frac{d}{dt}\left(\dot{r}_i \dot{r}_i r_j \alpha_j - r_i \dot{r}_i \dot{r}_j \alpha_j + \frac{g}{r} r_j \alpha_j\right)$$

$$= m \frac{d}{dt}\left((\dot{\vec{r}} \cdot \dot{\vec{r}}) \vec{r} - (\vec{r} \cdot \dot{\vec{r}}) \dot{\vec{r}} + \frac{g}{r} \vec{r}\right) \cdot \vec{\alpha}$$

Given that under this transformation the Lagrangian changes by a total time derivative, that is a symmetry of the system.

b) From Noether's theorem, the quantity

$$m \vec{R} \cdot \vec{\alpha} = \frac{\partial L}{\partial \dot{r}_i} \Delta r_i - m\left(\dot{r}_i \dot{r}_i r_j \alpha_j - r_i \dot{r}_i \dot{r}_j \alpha_j + \frac{g}{r} r_j \alpha_j\right)$$ is conserved.

$$\vec{R}\cdot\vec{\alpha} = \dot{r}_i \Delta r_i - \dot{r}_i \dot{r}_i r_j \alpha_j + r_i \dot{r}_i \dot{r}_j \alpha_j - \frac{g}{r} r_j \alpha_j$$

$$= \dot{r}_i \dot{r}_i r_j \alpha_j - r_i \dot{r}_i \dot{r}_j \alpha_j - \frac{g}{r} r_j \alpha_j$$

$$= \left((\dot{\vec{r}}\cdot\dot{\vec{r}})\vec{r} - (\vec{r}\cdot\dot{\vec{r}})\dot{\vec{r}} - \frac{g}{r}\vec{r} \right)\vec{\alpha}$$

Using the identity $\dot{\vec{r}} \times (\vec{r} \times \dot{\vec{r}}) = (\dot{\vec{r}}\cdot\dot{\vec{r}})\vec{r} - (\vec{r}\cdot\dot{\vec{r}})\dot{\vec{r}}$,

$$\vec{R} = \dot{\vec{r}} \times (\vec{r} \times \dot{\vec{r}}) - g\hat{r}$$

$$\boxed{\vec{R} = \dot{\vec{r}} \times \frac{\vec{L}}{m} - g\hat{r}}$$