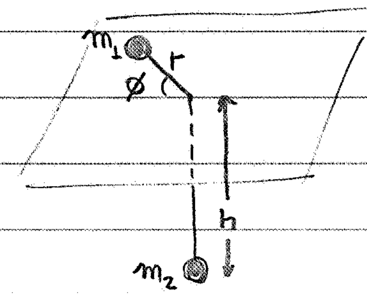
PHYSICS 210 - PROBLEM SET #2 - SOLUTIONS

PROBLEM 1



a) The Lagrangian is:

$$L = K - U = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 \dot{h}^2 + m_2 g h .$$

With $x_1 = r\cos\phi$, $y_1 = r\sin\phi$ and $h = L - r$ (where L is the length of the string):

$$\boxed{L = \frac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2 - m_2 g r}$$

The equations of motion are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} \Rightarrow \boxed{(m_1+m_2)\ddot{r} = m_1 r \dot{\phi}^2 - m_2 g}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} \Rightarrow \boxed{\frac{d}{dt}(m_1 r^2 \dot{\phi}) = 0}$$

b) The conserved angular momentum is $\boxed{L = m_1 r^2 \dot{\phi}}$.

The conserved energy is $\boxed{\frac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{m_1 r^2 \dot{\phi}^2}{2} + m_2 g r}$.

c) Uniform circular motion $\Rightarrow$ $\dot{\phi}$ = constant, r = constant

In this case $\dot{r} = \ddot{r} = 0$ and $m_1 r \dot{\phi}^2 - m_2 g = 0 \Rightarrow \boxed{L^2 = m_1 m_2 g r^3}$

Also, $E = \dfrac{L^2}{2m_1 r^2} + m_2 g r = \dfrac{3}{2} \dfrac{(m_2 g L)^{2/3}}{m_1^{1/3}}$

d) Since the applied force is along the string, the torque is zero and therefore L does not change.

Substituting $r_c + \delta r$ in the equation of motion for r:

$$(m_1 + m_2) \frac{d^2}{dt^2}(r_c + \delta r) = \frac{L^2}{m_1 (r_c + \delta r)^3} - m_2 g$$

$\Rightarrow \quad \delta \ddot{r} = \dfrac{L^2}{m_1 r_c^3}\left(1 - \dfrac{3\delta r}{r_c}\right) - m_2 g$. From (c), $\dfrac{L^2}{m_1 r_c^3} - m_2 g = 0$,

$$\Rightarrow \boxed{\delta \ddot{r} = -3 \left(\frac{m_1 m_2^4 g^4}{L^2}\right)^{1/3} \delta r}$$

The solution to the above equation is a periodic function with period $2\pi \sqrt{\dfrac{1}{3}\left(\dfrac{L^2}{m_1 m_2^4 g^4}\right)^{1/3}}$, and hence the trajectory's radius will oscillate with very small amplitude around the original circular trajectory.

$\boxed{\text{PROBLEM 2}}$

a) $$\boxed{\partial_\mu j^\mu = \frac{1}{c}\frac{\partial}{\partial t}(c\rho) + \frac{\partial}{\partial x^i} j^i = \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0}$$

b) $$F^{0i} = \partial^0 A^i - \partial^i A^0 = \frac{\partial A^i}{\partial t} + \frac{\partial \phi}{\partial x^i} = -E^i$$

$$F^{ij} = \partial^i A^j - \partial^j A^i = -\epsilon^{ijk} B^k$$

c) $$S[x^\mu] = -mc\int_0^T dt\left(\frac{dx^\mu}{dt}\frac{dx_\mu}{dt}\right)^{1/2} = -mc\int_0^T (dx^\mu dx_\mu)^{1/2} = -mc^2\int_0^T d\tau$$

Given that the action has dimensions of energy x time, m has dimensions of $\frac{\text{energy} \times \text{time}}{c^2 \times \text{time}}$ = mass.

$$S[x^\mu] = -mc\int_0^T dt\left[\frac{dx^\mu}{dt}\frac{dx_\mu}{dt}\right]^{1/2} = -mc\int_0^T dt\left[\frac{dx^\mu}{ds}\frac{ds}{dt}\frac{dx_\mu}{ds}\frac{ds}{dt}\right]^{1/2}$$

$$\boxed{= -mc\int dt\left|\frac{ds}{dt}\right|\left[\frac{dx^\mu}{ds}\frac{dx_\mu}{ds}\right]^{1/2} = -mc\int ds\left[\frac{dx^\mu}{ds}\frac{dx_\mu}{ds}\right]^{1/2}}$$

d) With the Lagrangian being $L = -mc\sqrt{\frac{dx_\mu}{ds}\frac{dx^\mu}{ds}}$,

the equations of motion are:

$$\frac{d}{ds}\left(\frac{\partial L}{\partial(\partial x^\mu/\partial s)}\right) = \frac{\partial L}{\partial x^\mu} \Rightarrow \frac{d}{ds}\left(\frac{-mc\, dx^\mu/ds}{\sqrt{\frac{dx^\mu}{ds}\frac{dx_\mu}{ds}}}\right) = 0$$

Defining $p^\mu = \frac{-mc \; dx^\mu/ds}{\sqrt{\frac{dx_\mu}{ds}\frac{dx^\mu}{ds}}} \Rightarrow \boxed{\frac{dp^\mu}{ds} = 0}$

Indeed, $p_\mu p^\mu = p^2 = (mc)^2$ and

$$p^0 = \frac{-mc^2 \, dt/ds}{\sqrt{\frac{dx_\mu}{ds}\frac{dx^\mu}{ds}}}, \quad \boxed{p^i = \frac{-mc \, dx^i/ds}{\sqrt{\frac{dx_\mu}{ds}\frac{dx^\mu}{ds}}} = -\frac{dx^i}{dt}\frac{mc \, dt/ds}{\sqrt{\frac{dx_\mu}{ds}\frac{dx^\mu}{ds}}} = \frac{dx^i}{dt}\frac{p^0}{c}}$$

e) The first term in the action was shown to be reparametrization invariant in part (c). As for the 2nd term:

$$-\frac{q}{c}\int ds \frac{dx^\mu}{ds} A_\mu(x(s)) = -\frac{q}{c}\int ds \frac{dx^\mu}{ds'}\frac{ds'}{ds} A_\mu(x(s')) = -\frac{q}{c}\int ds' \frac{dx^\mu}{ds'} A_\mu(x(s'))$$

The equations of motion are $\frac{d}{d\tau}\left(\frac{\partial L}{\partial(\partial x^\mu/\partial \tau)}\right) = \frac{\partial L}{\partial x^\mu}$

$$\Rightarrow \frac{d}{d\tau}\left(p_\mu - \frac{q}{c}A_\mu\right) = -\frac{q}{c}\frac{dx^\alpha}{d\tau}\frac{\partial A_\alpha}{\partial x^\mu}$$

$$\Rightarrow \frac{dp_\mu}{d\tau} - \frac{q}{c}\frac{\partial A_\mu}{\partial x^\alpha}\frac{dx^\alpha}{d\tau} = -\frac{q}{c}\frac{\partial A_\alpha}{\partial x^\mu}\frac{dx^\alpha}{d\tau} \Rightarrow \frac{dp_\mu}{d\tau} = \frac{q}{c}\left(\frac{\partial A_\mu}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\mu}\right)\frac{dx^\alpha}{d\tau}$$

$\Rightarrow \boxed{\frac{dp_\mu}{d\tau} = \frac{q}{c}F_{\mu\alpha}\frac{dx^\alpha}{d\tau}}$, i.e., the relativistic form of the Lorentz force equation.