Physics 210 - Problem Set #1 Solutions

Problem 1

(a) Rewritting (2) in terms of ρ:

$$\frac{d^2\rho}{d\phi^2} + \rho - \frac{g^2}{c^2} = 0$$

Multiplying by $\frac{d\rho}{d\phi}$ and integrating:

$$\frac{d\rho}{d\phi}\frac{d^2\rho}{d\phi^2} + \rho\frac{d\rho}{d\phi} - \frac{g^2}{c^2}\frac{d\rho}{d\phi} = \frac{d}{d\phi}\left(\frac{1}{2}\left(\frac{d\rho}{d\phi}\right)^2 + \frac{1}{2}\rho^2 - \frac{g^2}{c^2}\rho\right) = 0$$

$$\Rightarrow \quad \left(\frac{d\rho}{d\phi}\right)^2 + \rho^2 - \frac{2g^2}{c^2}\rho = K \qquad (*)$$

where K is a constant of integration.

In the apogee ρ_+ and perigee ρ_-, $d\rho/d\phi = 0$, implying that

$$\rho_+^2 - 2\frac{g^2}{c^2}\rho_+ = K \quad , \quad \rho_-^2 - 2\frac{g^2}{c^2}\rho_- = K$$

Solving for $2g^2/c^2$ and K, we obtain $\begin{cases} 2g^2/c^2 = \rho_+ + \rho_- \\ K = \rho_+\rho_- \end{cases}$

Replacing that in (*):

$$\left(\frac{d\rho}{d\phi}\right)^2 = -\rho^2 + (\rho_+ + \rho_-)\rho - \rho_+\rho_- = (\rho_+ - \rho)(\rho - \rho_-)$$

$$\Rightarrow \boxed{\frac{d\phi}{d\rho} = \frac{1}{\sqrt{(\rho_+ - \rho)(\rho - \rho_-)}}}$$

The separation between ρ_+ and ρ_- is:

$$\Delta\phi = \int_{\phi(\rho_-)}^{\phi(\rho_+)} d\phi = \int_{\rho_-}^{\rho_+} \frac{d\rho}{\sqrt{(\rho_+ - \rho)(\rho - \rho_-)}} = \operatorname{Tan}^{-1}\left(\frac{2\rho - \rho_+ - \rho_-}{2\sqrt{(\rho_+ - \rho)(\rho - \rho_-)}}\right)\Bigg|_{\rho_-}^{\rho_+}$$

$$= \operatorname{Tan}^{-1}(+\infty) - \operatorname{Tan}^{-1}(-\infty) = \boxed{\pi}$$

(b) From $\ddot{r} = -\frac{1}{m}\frac{dV}{dr} + \frac{c^2}{r^3}$ and $V(r) = mAr$, we obtain the analogue of (2):

$$-\frac{c^2}{r^2}\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) - \frac{c^2}{r^3} = -A$$

$$\Rightarrow \frac{d^2\rho}{d\phi^2} + \rho = \frac{A/c^2}{\rho^2}$$

Proceeding analogously:

$$\frac{d\rho}{d\phi}\frac{d^2\rho}{d\phi^2} + \rho\frac{d\rho}{d\phi} - \frac{A/c^2}{\rho^2}\frac{d\rho}{d\phi} = \frac{d}{d\phi}\left(\frac{1}{2}\left(\frac{d\rho}{d\phi}\right)^2 + \frac{1}{2}\rho^2 + \frac{A/c^2}{\rho}\right) = 0$$

$$\Rightarrow \left(\frac{d\rho}{d\phi}\right)^2 = -\rho^2 - \frac{2A/c^2}{\rho} + K$$, where K is a constant of integration.

In the apogee ρ_+ and perigee ρ_-, $d\rho/d\phi = 0$, implying that

$$-\rho_+^2 - \frac{2A/c^2}{\rho_+} + K = 0 \quad , \quad -\rho_-^2 - \frac{2A/c^2}{\rho_-} + K = 0 .$$

Solving for $2A/c^2$ and K, we obtain

$$\left(\frac{d\rho}{d\phi}\right)^2 = -\rho^2 - \frac{\rho_+\rho_-(\rho_+ + \rho_-)}{\rho} + \rho_+^2 + \rho_-^2 + \rho_+\rho_-$$

$$\Rightarrow \left(\frac{d\rho}{d\phi}\right)^2 = \frac{1}{\rho}(\rho_+ - \rho)(\rho - \rho_-)(\rho + \rho_+ + \rho_-)$$

$$\Rightarrow \boxed{\frac{d\phi}{d\rho} = \sqrt{\frac{\rho}{(\rho_+ - \rho)(\rho - \rho_-)(\rho + \rho_+ + \rho_-)}}}$$

Numerically integrating $\frac{d^2\rho}{d\phi^2} + \rho - \frac{1}{\rho^2} = 0$

with initial conditions $\rho = \frac{1}{2}$ and $\dot{\rho} = 0$, one obtains

that $\boxed{\Delta\phi = 1.76 \text{ radians}}$.

For an almost circular orbit, one solves the above equation with $\rho = 1 - \epsilon$, with $\epsilon \ll 1$. One obtains

$$\boxed{\Delta\phi = 1.81 \text{ radians} = \pi/\sqrt{3}}$$

For a very elongated orbit (i.e., $r_+ \to \infty$), one solves the equation with $\rho = \epsilon \ll 1$. One obtains

$$\boxed{\Delta\phi = \pi/2}$$

PROBLEM 2

(a) Inverting the relations (7), one obtains:

$$\boxed{\begin{aligned} \vec{r}_1 &= \vec{R} + \frac{m_2}{m_{12}}\vec{r} - \frac{m_3}{M}\vec{\rho} \\ \vec{r}_2 &= \vec{R} - \frac{m_1}{m_{12}}\vec{r} - \frac{m_3}{M}\vec{\rho} \\ \vec{r}_3 &= \vec{R} + \frac{m_{12}}{M}\vec{\rho} \end{aligned}}$$

(b) $T = \frac{1}{2} m_1 |\dot{\vec{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\vec{r}}_2|^2 + \frac{1}{2} m_3 |\dot{\vec{r}}_3|^2$

Replacing the relations obtained in (a), after a long and straightforward manipulation, we have:

$$T = \frac{1}{2} M |\dot{\vec{R}}|^2 + \frac{1}{2}\frac{m_1 m_2}{m_{12}} |\dot{\vec{r}}|^2 + \frac{1}{2}\frac{m_{12} m_3}{M} |\dot{\vec{\rho}}|^2$$

$$= \frac{1}{2} M |\dot{\vec{R}}|^2 + \frac{1}{2}\mu |\dot{\vec{r}}|^2 + \frac{1}{2}\mu_3 |\dot{\vec{\rho}}|^2$$

(c) $\vec{L} = m_1 \vec{r}_1 \times \dot{\vec{r}}_1 + m_2 \vec{r}_2 \times \dot{\vec{r}}_2 + m_3 \vec{r}_3 \times \dot{\vec{r}}_3$

Similarly, a straightforward substitution yields:

$$\boxed{\vec{L} = M \vec{R} \times \dot{\vec{R}} + \mu \vec{r} \times \dot{\vec{r}} + \mu_3 \vec{\rho} \times \dot{\vec{\rho}}}$$

(d) Using (7), (a) and

$$m_1 \ddot{\vec{r}}_1 = -\hat{r}_{12} F(r_{12}) - \hat{r}_{13} F(r_{13})$$
$$m_2 \ddot{\vec{r}}_2 = -\hat{r}_{21} F(r_{21}) - \hat{r}_{23} F(r_{23})$$
$$m_3 \ddot{\vec{r}}_3 = -\hat{r}_{31} F(r_{31}) - \hat{r}_{32} F(r_{32})$$

one obtains :

$$M\ddot{\vec{R}} = 0$$

$$\mu \ddot{\vec{r}} = -\hat{r} F(r) - \frac{m_2}{m_{12}} \frac{\left(\frac{m_2}{m_{12}}\vec{r} - \vec{\rho}\right)}{\left|\frac{m_2}{m_{12}}\vec{r} - \vec{\rho}\right|} F\left(\left|\frac{m_2}{m_{12}}\vec{r} - \vec{\rho}\right|\right) - \frac{m_1}{m_{12}} \frac{\left(\frac{m_1}{m_{12}}\vec{r} + \vec{\rho}\right)}{\left|\frac{m_1}{m_{12}}\vec{r} + \vec{\rho}\right|} F\left(\left|\frac{m_1}{m_{12}}\vec{r} + \vec{\rho}\right|\right)$$

$$\mu_3 \ddot{\vec{\rho}} = \frac{\left(\frac{m_2}{m_{12}}\vec{r} - \vec{\rho}\right)}{\left|\frac{m_2}{m_{12}}\vec{r} - \vec{\rho}\right|} F\left(\left|\frac{m_2}{m_{12}}\vec{r} - \vec{\rho}\right|\right) - \frac{\left(\frac{m_1}{m_{12}}\vec{r} + \vec{\rho}\right)}{\left|\frac{m_1}{m_{12}}\vec{r} + \vec{\rho}\right|} F\left(\left|\frac{m_1}{m_{12}}\vec{r} + \vec{\rho}\right|\right)$$