Physics 210 – Problem Set # 9

(due Thursday, December 2)

1. In a particle accelerator, charged particles move in approximately circular orbits under the influence of an external magnetic field. As long as the motion is nonrelativistic, the Hamiltonian governing the motion of a charged particle is

$$H = \frac{1}{2m}(\vec{p} - \frac{q}{c}\vec{A})^2 \ , \tag{1}$$

where $\vec{A}(\vec{x})$ is the external vector potential.

(a) Assume that $\vec{R}(t)$ is a solution to the equations of motion, that is, a fixed orbit (or 'reference orbit') in the accelerator. Let s be the distance measured along the path $\vec{R}(t)$. Then

$$\hat{\alpha}(s) = \frac{d\vec{R}}{ds} \tag{2}$$

is a unit vector along the path. Let the unit vector $\hat{\beta}(s)$ and the magnitude $\Omega(s)$ be defined by the equation

$$\frac{d\hat{\alpha}}{ds} = -\Omega(s)\hat{\beta}(s) \ . \tag{3}$$

Define $\hat{\gamma}(s)$ by the equation

$$\hat{\gamma}(s) = \hat{\alpha}(s) \times \hat{\beta}(s) \ . \tag{4}$$

Describe the relation of $\hat{\beta}$ and $\hat{\gamma}$ to the path $\vec{R}(s)$.

(b) Show that $d\hat{\gamma}/ds$ is orthogonal to $\hat{\alpha}$ and to $\hat{\gamma}$. Define $\omega(s)$ by the relation

$$\frac{d\hat{\gamma}}{ds} = -\omega(s)\hat{\beta}(s) \ . \tag{5}$$

Compute $d\hat{\beta}/ds$ in terms of these ingredients.

(c) It is convenient to choose as new coordinates the parameter s and two parameters measuring the distances orthogonal to the reference orbit. Let

$$\vec{x} = \vec{R}(s) + Y\hat{\beta} + Z\hat{\gamma} \ . \tag{6}$$

The generating function

$$F_2(\vec{p}, s, Y, Z) = \vec{p} \cdot [\vec{R}(s) + Y\hat{\beta} + Z\hat{\gamma}] \tag{7}$$

implements this change of variables as a canonical transformation. Construct the Hamiltonian in the new coordinate system, as a function of s, Y, Z and the conjugate momenta p_s, p_Y, p_Z.

(d) For constant $\vec{B} = B_0\hat{z}$, there is a circular cyclotron orbit with

$$\omega_C = \frac{qB_0}{mc} , \tag{8}$$

at a radius R_0 which depends on the energy through

$$R_0 = v/\omega_C . \tag{9}$$

For this circular orbit at radius R_0, compute the various quantities in part (a) explicitly and construct the Hamiltonian of part (c).

(e) Expand the Hamiltonian of part (e) for small displacements Y, Z, from the equilibrium orbit. Solve the resulting Hamiltonian equations of motion and analyze the stability of these small displacements.

(f) Now consider an inhomogeneous magnetic field. Show that the expression

$$\vec{B} = B_0 \left[\hat{z}(1 - n\frac{r - R_0}{R_0}) - n\hat{r}\frac{z}{R_0} \right] \tag{10}$$

respects the static Maxwell equation $\vec{\nabla} \times \vec{B} = 0$ to first order in small deviations.

(g) Repeat the analysis of part (e) for a magnetic field which has the form given in part (f) near the reference orbit. Show that there is a range of n for which the motion is stable with respect to both Y and Z oscillations.

2. Study the logistic map

$$x_{n+1} = 4\lambda x_n(1 - x_n)$$

a bit further.

(a) Show analytically that the period 2 cycle becomes unstable at the value of λ

$$\lambda_3 = \frac{1}{4}(1 + \sqrt{6}) = 0.86237$$

(b) Write a program to iterate the recursion numerically. Find the limiting behavior for some values of λ including

$$0.5 \ , \ 0.73 \ , \ 0.86 \ , \ 0.89 \ , \ 0.90 \ , \ 0.93 \ , \ 0.958 \ , \ 0.97$$