Physics 210 – Problem Set # 8

(due Thursday, November 18)

1. Consider a particle moving in 3 dimensions in a Coulomb potential.

   (a) Set up the problem in spherical coordinates and show that the Hamiltonian is

   $$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2 \sin^2\theta} - \frac{e^2}{r} \ . \tag{1}$$

   (b) Write the Hamilton-Jacobi differential equation for this system. Show that it can be solved by writing

   $$W(r,\theta,\phi) = W_r(r) + W_\theta(\theta) + W_\phi(\phi) \ , \tag{2}$$

   where, with constants α_r, α_θ, α_ϕ,

   $$\frac{dW_\phi}{d\phi} = \alpha_\phi \ , \qquad \left(\frac{dW_\theta}{d\theta}\right)^2 + \frac{\alpha_\phi^2}{\sin^2\theta} = \alpha_\theta^2 \ , \tag{3}$$

   and

   $$\left(\frac{dW_r}{dr}\right)^2 + \frac{\alpha_\theta^2}{r^2} = 2m\left(E + \frac{e^2}{r}\right) \ . \tag{4}$$

   (c) Solve these equations for the W_i and find integral expressions for the action variables J_r, J_θ, J_ϕ.

   (d) Evaluate these integrals explicitly and show that

   $$H = -\frac{me^4}{2(J_r + J_\theta + J_\phi)^2} \ . \tag{5}$$

   (e) Redo the analysis for the case of a relativistic (scalar) particle in a Coulomb potential, using as the Hamiltonian

   $$H = \sqrt{(pc)^2 + (mc^2)^2} - \frac{e^2}{r} \ , \tag{6}$$

   where c is the speed of light. This Hamiltonian includes the rest mass of the particle. The nonrelativistic Hamiltonian is recovered by expanding in p/mc, as follows:

   $$H = mc^2 + \frac{p^2}{2m} - \frac{e^2}{r} + \mathcal{O}(p^4) \ . \tag{7}$$

   In any event, show that the new Hamiltonian can be rewritten in terms of action-angle variables as

   $$\frac{H}{mc^2} = \left[1 + \frac{e^4}{(J_r c + [(J_\theta + J_\phi)^2 c^2 - e^4]^{1/2})^2}\right]^{-1/2} \ . \tag{8}$$

2. Numerically solve the Lorenz system of equations

$$\begin{aligned}
\dot{x} &= -10x + 10y && (9)\\
\dot{y} &= -y + rx - xz && (10)\\
\dot{z} &= -\frac{8}{3}z + xy && (11)
\end{aligned}$$

for various initial conditions and values of r to verify the behavior of this equation discussed in class, and in the supplement to Fetter and Walecka, chapter 9. In particular,

(a) With $r = 28$, integrate from the initial condition $(x, y, z) = (10, 10, 18)$, plot the resulting flow in the (x, z) plane, and verify Fig. 9.6 of Fetter and Walecka.

(b) Keeping $r = 28$, integrate the equation from some nearby initial conditions. Plot the resulting flows in the (x, z) and (x, y) planes, and plot the distance between the pairs of solutions as a function of time.