# Physics 210 – Problem Set # 6

(due Thursday, November 4)

1. A 'molecule' consists of three atoms of mass $m$ constrained to move in a plane. Write the potential energy of the three atoms as

$$V(|\vec{r}_1 - \vec{r}_2|) + V(|\vec{r}_2 - \vec{r}_3|) + \overline{V}(|\vec{r}_3 - \vec{r}_1|) \ . \qquad (1)$$

Assume that the total potential energy is minimized in the configuration in which the three atoms form an isoceles right triangle of (short) side $a$, with atom 2 at the right angle. Let

$$k = \left.\frac{d^2V}{dr^2}\right|_{r=a} , \qquad \overline{k} = \left.\frac{d^2\overline{V}}{dr^2}\right|_{r=\sqrt{2}a} \qquad (2)$$

This system has six eigenmodes of small oscillations. Identify the three modes with zero eigenvalues. Find the other three eigenmodes and their corresponding frequencies of oscillation.

In principle, you could solve this problem by asking Mathematica or Maple to diagonalize a $6 \times 6$ matrix. I do not recommend this route, but if you take it, sketch the forms of the six eigenmodes and explain how you could have solved this problem without having to explicitly diagonalize any matrix larger than $2 \times 2$.

[This is Fetter and Walecka's Problem 4.9.]

2. In this problem, we will compute the precession of the orbit of the Moon, obtaining results first derived by Newton. In carrying out this analysis, consider the Earth, Sun, and Moon as point masses with $m_S \gg m_E \gg m_M$. Let $\vec{r}_S$ be the vector from the Sun to the Earth, and let $\vec{r}$ be the vector from the Earth to the Moon. The orbit of the Earth around the Sun and the orbit of the Moon around the Earth are almost circular and coplanar. In that limit, the orbital angular velocities are

$$\begin{aligned} \text{Earth – Sun :} \qquad & n' = (G_N m_S / r_S^3)^{1/2} \\ \text{Moon – Earth :} \qquad & n = (G_N m_E / r^3)^{1/2} \end{aligned} \qquad (3)$$

To first approximation, ignore the influence of the Sun on the orbit of the Moon around the Earth. Then the Moon's orbit is an ellipse in a fixed plane. This orbit has a small eccentricity $e$, and there is a small angle $(\pi/2 - \Omega)$ between the orbit plane and the plane of the Earth and Sun, the plane of the ecliptic. (More precisely, $\Omega$ is the angle between the normal to the plane of the ecliptic and the line from the Earth to the perigee of the Moon's orbit.) The line of intersection of these two planes is called the 'line of nodes'. Let $h$ be the angle from a fixed direction in the plane of the ecliptic to the line of nodes. Let $g$ be the angle from the line of nodes to the perigee of the moon in the moon's orbital plane. In the limit $\Omega \approx \pi/2$, the two planes coincide. In this limit, we can denote the angle from the fixed direction to the perigee as $\omega = h + g$. Let $\vec{c} = \vec{r} \times \dot{\vec{r}}$ be the angular momentum/mass of the Moon, and let $\gamma = G_N m_E$.

(a) To take the influence of the Sun into account, it is simplest to work in a coordinate system in which the Earth is at rest and the Sun slowly makes a circular orbit around the Earth. For the Earth, the Sun's gravitational pull cancels the centrifugal force, and the Coriolis forces are negligible. For the Moon, the gravitational and centrifugal forces would cancel if the Moon were placed at the Earth's position, but if $\vec{r}$ is nonzero, there is a net 'tidal' force. Show that the total force of the Sun on the Moon in this frame is given by

$$\vec{F}/m_M = -(n')^2(\vec{r} - 3\hat{r}_S\hat{r}_S \cdot \vec{r}) \ , \tag{4}$$

where $\hat{r}_S$ is the unit vector in the direction of $\vec{r}_S$.

(b) We now consider the limit $\Omega \approx \pi/2$ and compute $\dot{\omega}$ for small but nonzero eccentricity $e$. This is most easily done by using the Runge-Lenz vector

$$\vec{R} = \dot{\vec{r}} \times \vec{c} - \gamma\hat{r} \tag{5}$$

Choose coordinates so that $\vec{R}$ points along the $\hat{x}$ axis and $\vec{c}$ points in the $\hat{z}$ direction. In these coordinates,

$$\vec{r} = \frac{c^2/\gamma}{1 + e\cos f}(\cos f, \sin f) \tag{6}$$

and $\hat{r}_S$ also lies in the $(\hat{x}, \hat{y})$ plane. Find $\dot{\vec{R}}$ in terms of $\vec{F}$ in part (a). Show that we can compute $\dot{\omega}$ from the formula

$$\dot{\vec{R}} \cdot \hat{y} = \gamma e \dot{\omega} \tag{7}$$

Find an expression for the left-hand side of this equation and average it over the lunar and solar orbital periods. Remember that

$$\int dt \dot{\vec{R}} \cdot \hat{y} = \int df (\frac{df}{dt})^{-1} \dot{\vec{R}} \cdot \hat{y} \ ; \tag{8}$$

the Jacobian adds a nontrivial contribution. Conclude that, over a time much longer than a year,

$$\frac{1}{n}\dot{\omega} = +\frac{3}{4}(\frac{n'}{n})^2 \ . \tag{9}$$

(c) Now set $e = 0$ but consider $\pi/2 - \Omega \neq 0$. Now we can compute $\dot{h}$. This is most easily done by computing the precession of $\vec{c}$. Choose coordinates so that the $(\hat{x}, \hat{y})$ plane is the plane of the ecliptic, with the line of nodes along $\hat{y}$. Show that, in these coordinates,

$$\vec{r} = \frac{c^2}{\gamma}(\cos f \sin \Omega, \sin f, -\cos f \cos \Omega) \ . \tag{10}$$

Show that we can compute $\dot{h}$ from the formula

$$\dot{\vec{c}} \cdot \hat{y} = c \cos \Omega \dot{h} \ . \tag{11}$$

Compute the left-hand side and average over the solar and lunar cycles. Conclude that, over a long period of time,

$$\frac{1}{n}\dot{h} = -\frac{3}{4}(\frac{n'}{n})^2 \ . \tag{12}$$