

Physics 210 – Problem Set # 5

(due Thursday, October 28)

1. Compute my inertia tensor. For simplicity, you can consider me to be composed of the simple elements listed below. Find the principal moments of inertia and the principal axes in each case.
 - (a) With arms at my sides, you can approximate me as a vertical cylinder of radius 15 cm, and height 180 cm, with density 1g/cm^3 . Compute the inertia tensor about the center of mass of this cylinder. Use this same reference point for the inertia tensor calculations in parts (b) and (c).
 - (b) Now add arms extended horizontally to the sides. These can be represented as two cylinder of the same density, length 80 cm, and radius 5 cm, pointing outward horizontally. They should be attached at the center of the end circle of the arms to points 20 cm below the top of the vertical cylinder.
 - (c) Finally, move the arms so that they are pointing horizontally forward. Attach them to the same points on the vertical cylinder at points on the rim of the end circles.

2. I told you in class that a completely asymmetrical top rotates stably about the axes with the largest and smallest moments of inertia but that it is unstable when rotating about the axis with the intermediate moment of inertia. Investigate the motion in the unstable case by numerically integrating Euler's equations. Take, for example, $I_1 = 1$, $I_2 = 2$, and $I_3 = 4$, $\Omega = 2$, and $\epsilon = 0.1$ in the following examples. Follow the motion up to about $t = 10$.
 - (a) Consider the situation $\omega_3 = \Omega$. Choose ω_1 and ω_2 to have various values of order ϵ . Graph ω_1 versus t and ω_2 versus t .
 - (b) Consider the situation $\omega_1 = \Omega$. Choose ω_2 and ω_3 to have various values of order ϵ . Graph ω_2 versus t and ω_3 versus t .
 - (c) Consider the situation $\omega_2 = \Omega$. Choose ω_1 and ω_3 to have various values of order ϵ . Graph ω_1 versus t and ω_3 versus t .
 - (d) Explain qualitatively the form of the solutions in part (c), using conservation of energy and angular momentum.

3. A gyroscope in the form of a symmetric top is mounted at its center of mass, so that it receives no gravitational torque, and is constrained to move in a horizontal plane. The gyroscope is set spinning about its axis of symmetry ($\hat{3}$ axis) with angular velocity Ω .

- (a) Let ω be the angular velocity of the earth's rotation, and assume $\omega \ll \Omega$. The Coriolis force exerts the following torque on the gyroscope about its center of mass:

$$\vec{\tau} = -2 \int d^3r \rho(r) [\vec{r} \times (\vec{\omega} \times (\vec{\Omega} \times \vec{r}))] \quad (1)$$

Derive this expression, and evaluate it in terms of the inertia tensor of the top.

- (b) Show that the gyroscope will oscillate about the northerly direction. If the gyroscope is located at a latitude θ , and ϕ is the angle from the north, show that

$$\ddot{\phi} = - \left(\frac{I_3}{I_1} \Omega \omega \cos \theta \right) \sin \phi \quad (2)$$

[This is Problem 5.2 from Fetter and Walecka's book.]

4. Consider a space ship moving in deep space away from any gravitational fields. The ship has the form of a spherical shell of radius a , cut in half by a heavy circular metal plate, of the same radius, which provides a floor. Take the center of the sphere to be initially at $\vec{r} = 0$, the floor to be in the (\hat{x}, \hat{y}) plane, and the shell to be rotating about the \hat{z} axis with angular velocity Ω . Take the mass of the floor to be M , and ignore the mass of the rest of the shell. At the center of the sphere, imagine that there is a monkey of mass $m = \frac{1}{4}M$. You can approximate the monkey as a point mass.

- (a) The monkey begins to walk slowly outward along the floor. Because we are in deep space and momentum is conserved, the center-of-mass of the total system remains fixed. Compute the distance of the center of the shell from the center of mass, as a function of the radius r that the monkey has reached.
- (b) Compute the inertia tensor of the system about its center of mass, as a function of the monkey's position r . Using this result, compute the angular velocity of the ship as a function of r .
- (c) After the monkey reaches the wall of the shell, he begins to climb up along the inside of the shell toward the north pole. Let θ be the monkey's polar angle, with $\theta = \pi/2$ when the monkey is on the floor and $\theta = 0$ when the monkey is at the top. Find the distance between the center of mass of the ship and the center of the shell, as a function of θ .
- (d) Compute the inertia tensor of the system about its center of mass, in a frame fixed in the shell, as a function of θ .
- (e) The energy of rotation of the system is given by

$$E = \frac{1}{2} \omega_i I_{ij} \omega_j = \frac{1}{2} L_i (I^{-1})_{ij} L_j . \quad (3)$$

As the monkey climbs, \vec{L} is conserved. If the monkey climbs adiabatically (in other words, using the least possible effort), he will supply (or remove) just enough

energy to stay in the minimum of this expression for fixed \vec{L} . Using this principle, find the orientation of the spaceship with respect to the \hat{z} axis as a function of θ . Does the spaceship end up rotating about the \hat{z} axis when the monkey reaches the top?