Physics 210 – Problem Set # 4

(due Thursday, October 21)

1. Consider a system of particles of charge q and mass m which move in a uniform magnetic field $\vec{B}$.

   (a) Show that this constant magnetic field is represented by the vector potential

   $$\vec{A} = -\frac{1}{2}\vec{r} \times \vec{B} \tag{1}$$

   (b) Write the Lagrangian for this system of particles. Show that the effect of $\vec{B}$ is removed, to first order in $\vec{B}$, by working in a coordinate system which rotates with the angular velocity

   $$\vec{\Omega}_L = -\frac{q}{2mc}\vec{B} \tag{2}$$

   This result is called *Larmor's theorem.*

2. Let $(R_{\hat{n}}(\alpha))_{ij}$ be the 3×3 matrix which implements a rotation by α about the axis $\hat{n}$.

   (a) Show that

   $$\chi(\alpha) = \mathrm{tr}[R_{\hat{n}}(\alpha)] = 1 + 2\cos\alpha \tag{3}$$

   This quantity is called the *character* of the rotation. To prove this relation, first consider a rotation about $\hat{n} = \hat{z}$, and then show that χ is independent of $\hat{n}$.

   (b) Use this relation to compute the rotation angle α in terms of the Euler angles. Show that

   $$\cos\frac{\alpha}{2} = \cos\frac{\phi + \psi}{2}\,\cos\frac{\theta}{2} \tag{4}$$

3. Consider a jet plane moving through the air at fixed speed v. Define a fixed coordinate system in the air by taking $\hat{y}$ to point upward and $\hat{z}, \hat{x}$ to be horizontal. We can also associate a coordinate system with the plane as follows: Let the $\hat{\zeta}$ axis point forward, let the $\hat{\xi}$ axis point toward the left wing, and let the $\hat{\eta}$ axis point up. Then we can describe the orientation of the jet with respect to the fixed coordinate system $(\hat{x}, \hat{y}, \hat{z})$ in terms of Euler angles.

   (a) Assuming that the jet is always instantaneously moving forward, that is, in the $\hat{\zeta}$ direction, show that its velocity vector is

   $$\vec{v} = v(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{5}$$

(b) Show that, with these coordinates, a banked turn is described by setting

$$\phi = 0 \ , \qquad \dot{\theta} = \omega \ , \qquad \psi = (\text{const}) \tag{6}$$

with ω constant. If the initial velocity is in the $\hat{z}$ direction, find the velocity and the position of the jet as a function of time.

(c) The situation

$$\phi = (\text{const}) \ , \qquad \dot{\theta} = \omega \ , \qquad \psi = 0 \tag{7}$$

leads to a different outcome. Find the velocity and position as a function of time in this case.

(d) Now consider a situation in which the pilot maintains a fixed angular velocity

$$\vec{\Omega} = a\hat{\zeta} + b\hat{\eta} \tag{8}$$

relative to the jet. Convert this expression to three first-order differential equations for the Euler angles by considering the relation

$$\begin{aligned} \vec{\Omega} \ &= \ \hat{\xi}[-\dot{\phi}\sin\theta\cos\psi + \dot{\theta}\sin\psi] + \hat{\eta}[\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi] \\ &\quad + \hat{\zeta}[\dot{\phi}\cos\theta + \dot{\psi}] \end{aligned} \tag{9}$$

and solving for $\dot{\theta}$, $\dot{\phi}$, $\dot{\psi}$.

(e) Initially, $\theta = \phi = \psi = 0$. Analyze the equations for small t starting from this initial condition. Show that

$$\theta = bt \ , \qquad \phi = \psi = \frac{1}{2}at \ , \tag{10}$$

to first order in t.

(f) Since the equations obtained in part (d) are singular when $\theta, \psi = 0$, it is not completely straightforward to integrate them numerically from $t = 0$. To evade this problem, use the result of part (e) for small times, and integrate numerically starting from $t = \epsilon$ (with, for example, $\epsilon = 0.001$). Using this strategy, study the case $a = 1/\text{sec}$, $b = 2/\text{sec}$, and find the three Euler angles numerically as a function of time.

(g) Now let $v = 500$ km/sec. Find, numerically, the height of the jet relative to its original trajectory as a function of time. Follow the motion from 0 to 10 seconds.