Physics 210 – Problem Set # 3

(due Thursday, October 14)

1. Consider a point mass that moves without friction on a circular wire of radius a. Place the wire in a vertical orientation, with gravity acting downward on the mass. Rotate the wire around its vertical diameter with angular velocity Ω.

   (a) Construct the Lagrangian for the point mass, using the angle θ, measured from the bottom of the circle, as the generalized coordinate.

   (b) Show that the condition for the mass to remain at a single value $\theta = \theta_0$, executing circular motion, is

   $$\cos\theta_0 = \frac{g}{a\Omega^2} \ . \tag{1}$$

   (c) Compute the frequency of the small oscillation about this uniform circular motion.

   (d) Redo the analysis for the case in which the point mass moves on a straight wire, at a fixed angle θ_0 to the vertical, which rotates about the vertical axis with angular velocity Ω. Use ℓ, the displacement along the wire, as the generalized coordinate. Show that the condition for an equilibrium orbit is

   $$\ell_0 = \frac{g\cos\theta_0}{(\Omega\sin\theta_0)^2} \ . \tag{2}$$

   Find the behavior of the small displacements from this orbit.

   Note: This is problem 3.1 and 3.2 from Fetter and Walecka's book.

2. Consider once again the Lagrangian of the Kepler or hydrogen atom problem:

   $$L = \frac{1}{2}m|\dot{\vec{r}}|^2 + \frac{mg}{r} \tag{3}$$

   (a) Show that the following infinitesimal transformation, proportional to a vector parameter $\vec{\alpha}$, is a symmetry of this Lagrangian system:

   $$\Delta\vec{r} = 2\dot{\vec{r}}\,\vec{r}\cdot\vec{\alpha} - \vec{\alpha}\,\vec{r}\cdot\dot{\vec{r}} - \vec{r}\,\dot{\vec{r}}\cdot\vec{\alpha} \ . \tag{4}$$

   (b) Find the vector quantity which is conserved by virtue of this symmetry, and show that it is proportional to the Runge-Lenz vector:

   $$\vec{R} = \frac{1}{m}\dot{\vec{r}}\times\vec{L} - g\hat{r} \ . \tag{5}$$