

Physics 210 – Problem Set # 2

(due Thursday, Oct. 7)

1. A mass m_1 moves frictionlessly on a horizontal table. This mass is connected by a massless string through a small hole in the table to a mass m_2 which hangs vertically.
 - (a) Construct the Lagrangian for this system using the polar coordinates (r, ϕ) of m_1 , and find the equations of motion.
 - (b) By integrating these equations, show that this system has a conserved energy and angular momentum.
 - (c) Find the condition under which the mass m_1 executes uniform circular motion about the hole.
 - (d) If the uniform circular motion is perturbed by pulling down on m_2 suddenly and then letting go, what happens? Analyze the subsequent motion quantitatively, assuming that the perturbation is small.

2. This problem concerns the motion of relativistic particles. I denote a 4-vector by $v^\mu = (v^0, \vec{v})$ or $v_\mu = (v^0, -\vec{v})$. Then the invariant length of v is written $v^2 = v^\mu v_\mu = (v^0)^2 - (\vec{v})^2$. I denote a position in space-time by $x^\mu = (ct, \vec{x})$, where c is the speed of light. Then $x^2 = (ct)^2 - (\vec{x})^2$. For $x^2 > 0$, $x^2 = c^2\tau^2$, where τ is the ‘proper time’, the time measured by a clock that moves from 0 to x . Since

$$t \frac{\partial}{\partial t} + \vec{x} \cdot \frac{\partial}{\partial \vec{x}} \tag{1}$$

is a Lorentz-invariant quantity, one must write the derivative with respect to x^μ as

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial ct}, \frac{\partial}{\partial x^i} \right), \quad \partial^\mu = \left(\frac{\partial}{\partial ct}, -\frac{\partial}{\partial x^i} \right). \tag{2}$$

Then the object in eq. (1) is $x^\mu \partial_\mu$, and it looks Lorentz invariant.

- (a) Show that, if we write $j^\mu = (c\rho, \vec{j})$, then the equation $\partial_\mu j^\mu = 0$ is the usual continuity equation.
- (b) Let ϕ be the electromagnetic vector potential and let \vec{A} be the electromagnetic vector potential. These combine into a 4-vector $A^\mu = (\phi, \vec{A})$. Let

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{3}$$

Show that the tensor $F^{\mu\nu}$ contains the electric and magnetic fields in the following way: For $i, j, k = 1, 2, 3$ (spacelike directions),

$$F^{0i} = -E^i, \quad F^{ij} = -\epsilon^{ijk} B^k. \tag{4}$$

- (c) Let $x^\mu(t) = (ct, \vec{x}(t))$ be a path in space-time from x_a^μ to x_b^μ , in time T . An inspired idea is to take the action for the particle's motion to be proportional to the proper time elapsed as the particle moves along the path. That is

$$S[x^\mu] = -mc \int_0^T dt \left[\frac{dx^\mu}{dt} \frac{dx_\mu}{dt} \right]^{1/2} \quad (5)$$

Show that m has the dimensions of mass. Show that, if s is an arbitrary coordinate that increases along the path (so that we consider t and x^i to be, on an equal footing, functions of s), then the action given in eq. (5) is equal to the following expression:

$$S[x^\mu] = -mc \int ds \left[\frac{dx^\mu}{ds} \frac{dx_\mu}{ds} \right]^{1/2} \quad (6)$$

Thus, t does not play a special role. The property that any parameter s gives an equivalent action is called *reparametrization invariance*.

- (d) Find the variational equations of motion which follow from eq. (6). Show that they have the form

$$\frac{d}{ds} p^\mu = 0 \quad (7)$$

where

$$p^2 = (mc)^2 \quad \text{and} \quad p^i = \frac{p^0}{c} \cdot \frac{dx^i}{dt} \quad (8)$$

Thus p^μ is the energy-momentum 4-vector: $p^\mu = (E/c, \vec{p})$.

- (e) We can couple the relativistic particle to an electromagnetic field by writing

$$S[x^\mu] = -mc \int_0^T ds \left[\frac{dx^\mu}{ds} \frac{dx_\mu}{ds} \right]^{1/2} - \frac{q}{c} \int ds \frac{dx^\mu}{ds} A_\mu(x(s)) . \quad (9)$$

Show that this action is reparametrization invariant. Vary it with respect to $x^\mu(s)$ and show that it leads to the relativistic form of the Lorentz force equation.