Physics 210 – Problem Set # 1

(due Thursday, Sept. 30)

Many of the problems that I will give in this course will involve numerical operations such as the integration of differential equations and the diagonalization of matrices. These operations are easily accomplished by the use of modern mathematical analysis programs such as Matlab, Mathematica, and Maple. These programs are available to you on leland.stanford.edu, and most of you have at least one of these running on your personal computer. Problem 1 of this problem set will give you a chance to practice your numerical skills.

1. In class, we saw that, for a body of mass m orbiting a center of force and bound by a potential
$$V(r) = -m\frac{g}{r} \tag{1}$$
the orbit is given as the solution $r(\phi)$ to the equation
$$-\frac{c^2}{r^2}\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) - \frac{c^2}{r^3} = -\frac{g^2}{r^2} \tag{2}$$
where $c = L/m$. To solve this equation, we set $\rho = 1/r$; then ρ obeys the equation of a particle in a harmonic oscillator potential.

   (a) On the orbit, let r_- be the minimum value of r ('perigee') and r_+ be the maximum value of r ('apogee'), and let $\rho_- = 1/r_-$, $\rho_+ = 1/r_+$. Show that that
$$\frac{d\phi}{d\rho} = \frac{1}{\sqrt{(\rho_- - \rho)(\rho - \rho_+)}} \ , \tag{3}$$
and use this equation to show that r_- and r_+ are separated by $\Delta\phi = \pi$.

   (b) For the potential
$$V(r) = mA|r| \ , \tag{4}$$
derive the analogue of eqs. (2) and (3). Setting $A/c^2 = 1$, numerically integrate the equation for $d^2\rho/d\phi^2$ with the initial conditions $\rho = \frac{1}{2}$, $\dot{\rho} = 0$ to find the angular separation $\Delta\phi$ of the apogee and the perigee.

   (c) What happens to $\Delta\phi$ when $r_+ \approx 1$, an almost circular orbit? What happens when r_+ becomes large, so that the orbit is more and more elongated? (A purely numerical solution is sufficient, but if you wish, you can address this question analytically using the method of part (a).)

(d) Check the answer of part (b) by numerically integrating Newton's equations for this potential,

$$\frac{d^2\vec{r}}{dt^2} = -\hat{r} \tag{5}$$

with initial conditions $\vec{r} = (2, 0)$, $\dot{\vec{r}} = (0, \frac{1}{2})$ and measuring the angle between the apogee and the perigee.

2. Consider the 3-body problem with central forces. Jacobi proposed a useful set of coordinates which decouple the internal and center-of-mass motions. Let

$$M = m_1 + m_2 + m_3 \ , \qquad m_{12} = m_1 + m_2 \tag{6}$$

Then define

$$\begin{aligned} \vec{R} &= \frac{m_1}{M}\vec{r}_1 + \frac{m_2}{M}\vec{r}_2 + \frac{m_3}{M}\vec{r}_3 \\ \vec{r} &= \vec{r}_1 - \vec{r}_2 \\ \vec{\rho} &= \vec{r}_3 - \left(\frac{m_1}{m_{12}}\vec{r}_1 + \frac{m_2}{m_{12}}\vec{r}_2\right) \ . \end{aligned} \tag{7}$$

(a) Solve for $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$ in terms of $\vec{R}$, $\vec{r}$, $\vec{\rho}$.

(b) Show that the kinetic energy

$$T = \sum_i \frac{1}{2} m_i |\dot{\vec{r}}_i|^2 \tag{8}$$

takes a simple form in these coordinates:

$$T = \frac{1}{2}M|\dot{\vec{R}}|^2 + \frac{1}{2}\mu|\dot{\vec{r}}|^2 + \frac{1}{2}\mu_3|\dot{\vec{\rho}}|^2 \tag{9}$$

where

$$\mu = \frac{m_1 m_2}{m_{12}} \ , \qquad \mu_3 = \frac{m_{12} m_3}{M} \tag{10}$$

(c) Show that the total angular momentum $\vec{L}$ takes an equally simple form.

(d) Find the Newton equations for $\vec{R}$, $\vec{r}$, and $\vec{\rho}$.