Physics 210 – Final Exam

Each problem is worth 25 points, for a total of 100 points.

1. Consider a flat circular disk of mass M and radius a with a small linear channel along its diameter. A small body with mass M moves frictionlessly in this channel. This body is connected to the center of the disk by a spring of tension k, so that, if the position of the body is r, the restoring force is kr. The disk is constrained to rotate in its own plane about its center.

   (a) Show that the equations of motion for this system reduce to a single second-order differential equation for r. Write this equation.

   (b) Find the conditions under which there is a solution with r constant. Write the criterion for this solution to be at $r = 0$, and find the equilibrium value of r when this value is not at zero.

   (c) Find the frequency of small oscillations about the equilibrium in each case.

2. Consider a string of N beads, connected by springs with spring constant k, wrapping around a cylinder. Wrap another such string of N beads around the cylinder. Join the pairs of adjacent beads by springs with spring constant k, to form a ladder of beads wrapping around the cylinder.

   (a) Find the two zero-frequency modes of oscillation of this ladder.

   (b) The ladder has a relatively simple mode of oscillation in which the two strings of beads move rigidly toward and away from one another. Find the frequency of this oscillation.

   (c) The ladder has additional relatively simple modes of oscillation in which the beads of each pair remain at their equilibrium separation and oscillate together around the cylinder. There are N modes of this type (including one of the zero-frequency modes). Find them.

   (d) Find the remaining $(3N - 2)$ normal modes of this system and the corresponding frequencies.

3. Consider a moon interacting tidally with a planet. Model the tidal interaction of the earth and the moon by considering the moon as a point mass M_M moving in the (x, y) plane and the earth as a point mass M_E at $\vec{x} = 0$ with a tidal bulge, represented by a two masses m placed at the positions $\pm r_E$ along the $\hat{x}$ axis. But, unlike the real case, consider the earth's tidal bulge as fixed in space as the moon orbits.

(a) Show that the tidal force of the earth on the moon is given by

$$\vec{F} = \frac{2G_N m M_M r_E^2}{R^4}(3\hat{x}\hat{x}\cdot\hat{R} + \frac{3}{2}\hat{R} - \frac{15}{2}\hat{R}(\hat{R}\cdot\hat{x})^2) \ . \tag{1}$$

(b) Start the moon in an elliptical orbit aligned with the $\hat{x}$ axis:

$$r = \frac{c^2/g}{1 + ecos\phi} \ , \tag{2}$$

where $g = G_N m_E$, $c = L/m_M$, and e is the eccentricity of the orbit. Compute $\dot{c}$ and $\dot{e}$.

(c) What happens to the moon's orbit over a long period of time?

4. Consider a linear magnetic transport line, in which particles move under the Hamiltonian

$$H = \frac{1}{2m}(\vec{p} - \frac{q}{c}\vec{A})^2 \ . \tag{3}$$

The line $x = y = 0$ will be an equilibrium with $\vec{B} = 0$. Consider motions with p_z large and fixed and treat x, y, p_x, p_y as small perturbations.

(a) Consider a magnetic quadrupole field, given by the vector potential

$$A_z = \frac{B_0}{2R_0}(x^2 - y^2) \ . \tag{4}$$

Show that, in this field, the x and y degrees of freedom have independent oscillations. Find the solution to the equations of motion for x, p_x in the following form:

$$\begin{pmatrix} x(t) \\ p_x(t) \end{pmatrix} = A(t) \begin{pmatrix} x(0) \\ p_x(0) \end{pmatrix} \ , \tag{5}$$

where $A(t)$ is a 2×2 matrix. Show that $\det A = 1$. Why must this be true? Find the solution to the equations of motion for y, p_y in a similar form

$$\begin{pmatrix} y(t) \\ p_y(t) \end{pmatrix} = B(t) \begin{pmatrix} y(0) \\ p_y(0) \end{pmatrix} \ . \tag{6}$$

(b) Now consider a set of two magnetic elements of the same length, one of which contains the quadrupole field in part (a) and the other of which contains the quadrupole field with the opposite polarity

$$A_z = -\frac{B_0}{2R_0}(x^2 - y^2) \ . \tag{7}$$

Find a condition on the length, or on the time t for traversing each element, such that the combination of two elements is focussing both for x and for y. Show that, under this condition, it is necessarily defocussing for p_x and p_y.