Physics 210 – Final Exam

This exam is due Friday, December 10, at noon. Please hand it in Tomas Rube, the officemate of Daniele Alves, in Varian 363. Tomas will be there from 11 to 12 to collect the exams. If you have any questions about the exam, please contact Michael Peskin by email at the address mpeskin@slac.stanford.edu. If errata are reported, he will announce them on the course web site.

Do not collaborate on this exam. Return the exam in a blue book (or multiple blue books) with the honor code acknowledgement signed. Alternatively, copy the Stanford honor code from

http://www.stanford.edu/dept/vpsa/judicialaffairs/guiding/pdf/honorcode.pdf ,

sign it, and attach it to your exam solution.

The exam is worth a total of 100 points.

The exam is open-book. However, if you make strong use of a reference other than the class textbooks and notes, please cite the reference in your solutions. The exam can be worked using the course materials only. You will learn more if you do not try to find the answers in the literature.

1. (30 points) Analyze the set of equations

$$\dot{v} = \alpha v - \beta v^3 \tag{1}$$
$$\dot{T} = -T + \gamma v^2 \tag{2}$$

with $\alpha, \beta, \gamma > 0$, describing the motion of a convecting fluid

   (a) Where are the fixed points?

   (b) Study the linearized behavior at each fixed point, compute the stability eigenvalues, and draw the local flows. Which fixed points are stable?

   (c) Draw the qualitative form of the flows in the whole plane.

2. (30 points) Consider an assemblage of three equal mass points 1,2,3 arranged in a triangle. The points are constrained to lie in the $(\hat{x}, \hat{y})$ plane. They are connected by rubber bands, so that the potential between two points i, j is

$$\frac{1}{2}k(\vec{x}_i - \vec{x}_j)^2$$

This pulls the points inward toward one another. At the same time, a stick inserted into the triangle exerts a force of constant magnitude F directed outward from the midpoint of the segment between points 1 and 2 to the vertex 3. Similarly, an outward force F acts on the vertex 1 from the midpoint of the segment between the vertices 2 and 3 and on the vertex 2 from the midpoint of the segment between the vertices 3 and 1.

(a) Show that the equilibrium configuration is an equilateral triangle with side $a = F/\sqrt{3}k$.

(b) Show that the triangle is stable under the deformation in which the triangle remains equilateral and a increases or decreases.

(c) Find three modes of deformation of the triangle with neutral stability, that is, zero stability eigenvalues.

(d) We started with 6 orthogonal motions of the three points, and we have now accounted to 4 linear combinations of these. Construct the other two linear combinations and compute their stability matrix. Is this triangle construction stable, or not?

3. (40 points) Consider the interaction of a large planet such as Jupiter with an asteriod at a smaller radius from the sun. Use the approximation that the mass of Jupiter is much smaller than the mass of the sun, so that it provides a small perturbation on the orbit of the asteroid. Assume that Jupiter moves in a circular orbit and that the asteroid moves in an orbit that is close to a circle in the same plane. Let the distance between Jupiter and the sun be R, and let the distance from the asteriod to the sun be close to r.

   (a) Consider first the case in which the orbital period of Jupiter is incommensurate with that of the asteroid. In that case, over a long time, the asteroid sees an average of the gravitational potential set up by Jupiter over all positions of Jupiter on its circular orbit. You can approximate this potential near the radius r_0 of a circular asteriod orbit as

   $$V = A + B(r - r_0)$$

   Make an estimate of A and B. Then, working to first order in the mass ratio $M_J/M_\odot$, and assuming a circular orbit for the asteriod, compute the shift in its orbital period. If the orbit of the asteriod has a small ellipticity, the perihelion of the ellipse should precess. Compute the angular velocity of the precession.

   (b) Explain, in general terms, why this approximation breaks down if the orbital periods of Jupiter and the asteriod are in the relation of a ratio of integers.

   (c) Consider in particular the case in which the orbital periods of the asteriod and Jupiter are in a 1 : 2 ratio. In this case, if the orbit of the asteriod is elliptical, the perihelion of the orbit continually returns to the same orientation with respect to Jupiter. Sketch the unperturbed orbit of the asteriod in the frame that is rotating with Jupiter.

   (d) Let $r(\phi)$ be the distance of the asteriod from the sun, with ϕ measured from the orientation of Jupiter. Working in the rotating frame in which Jupiter is at rest, find an evolution equation for $\Delta r(\phi)$, the deviation of the asteriod from a circular orbit, including terms of first order in the mass of Jupiter.

   (e) Study the solutions of this equation for different orientations of the original elliptical orbit. For which orientations is the orbit stable?