

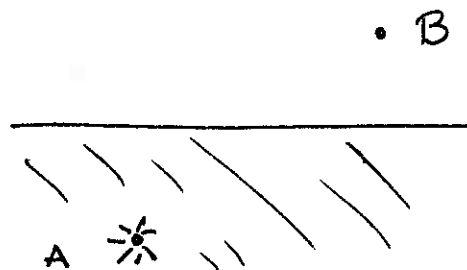
## Variational Principles

In the previous two lectures, we discussed many elementary aspects of mechanics that can be derived by integrating Newton's equations of motion. One can go a long way with these tools. Newton, after all, derived the theory of the tides and perturbations of the lunar orbit. However, I will now interrupt that development to introduce some more powerful formal methods. In the next three lectures, I will discuss the idea of Lagrange that equations of motion can be derived from a *variational principle*.

In this lecture, I will explain what a variational principle is and how one works with variational principles mathematically, a subject called *calculus of variations*. Variational principles are used in many areas of physics outside of particle mechanics. I will give some examples.

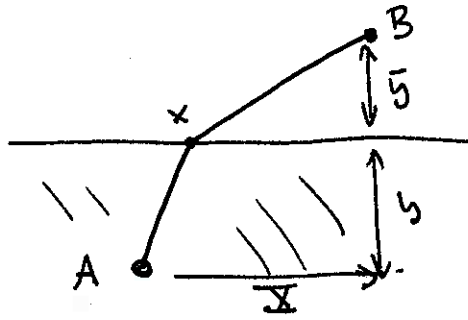
One of the problems that introduced variation calculus to physics is the question of the path of a light beam through a dielectric medium. Fermat formulated this problem as the *principle of least time*, the statement that light follows the path through the medium that requires the minimum amount of time. I will begin by analyzing this principle, beginning with a special case, then considering progressively more general situations.

Consider first a situation in which light travels from point A to point B, with B in vacuum (or air) and A in a medium with index of refraction  $n$ .



Our intuition would say that light should travel along a straight line which in the lower medium and also in a straight line through the upper region. However, it is not clear how or where to join these two straight lines. Fermat's principle gives a prescription for this.

If the light path runs in a straight line from A to a point  $x$  on the surface of the lower medium and then onward in a straight line to B, we can find the value of  $x$  that gives the least time for the complete path.



In the upper medium, the path length is

$$\bar{l} = [y^2 + (X-x)^2]^{\frac{1}{2}}$$

and the time is  $\bar{l}/c$ . IN the lower medium, the path length is

$$l = [y^2 + x^2]^{\frac{1}{2}}$$

and the time is  $ln/c$ . The total elapsed time is

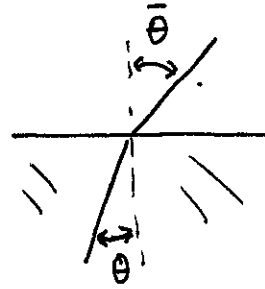
$$T = \frac{n}{c} [x^2 + y^2]^{\frac{1}{2}} + \frac{1}{c} [(X-x)^2 + 5^2]^{\frac{1}{2}}$$

Fermat's principle says that the  $x$  that is chosen is the one that minimizes this expression.

$$\frac{\partial T}{\partial x} = \frac{n}{c} \frac{x}{[x^2 + y^2]^{\frac{1}{2}}} - \frac{1}{c} \frac{(X-x)}{[(X-x)^2 + 5^2]^{\frac{1}{2}}}$$

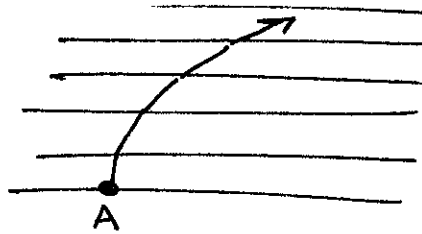
This gives exactly

$$n \sin \theta = \sin \bar{\theta}$$



that is, Snell's law! The paths adjust so that path is slightly shorter in the medium and longer in air, even at the expense of making the complete path longer, so that the total time is minimized.

At this point, Fermat's principle seems to be just an elegant reformulation of Snell's law. Its advantage is seen when we apply the principle to a more complicated problem. Consider a problem in which the medium is not a series of discrete elements with different indices of refraction but, rather, has a smooth variation from point to point  $n(y)$ . If the only tool we have at our disposal is Snell's law, we must divide the medium into small slices of approximately constant  $n$  and apply Snell's law at each interface.



It is much easier to approach this problem using the principle of least time. Begin with the similar idea of dividing the medium into slices of fixed  $n_i$ , and write the expression for the total time in terms of the positions  $x_i$  at which the light beam crosses each interface.

$$T = \sum_i \frac{n_i}{c} [(\Delta y)^2 + (x_i - x_{i-1})^2]^{\frac{1}{2}}$$

The principle of least time says that we should minimize this expression with respect to each of the  $x_i$ . We could do this directly, but actually it is easier to first pass to the limit in which the variation of  $n(y)$  is continuous (which is, after all, the problem that we really want to solve).

Instead of parametrizing the horizontal position by the slice  $i$ , we can parametrize

this position as a continuous function  $x(y)$ . The path length through a slice of small thickness  $\Delta y$  is

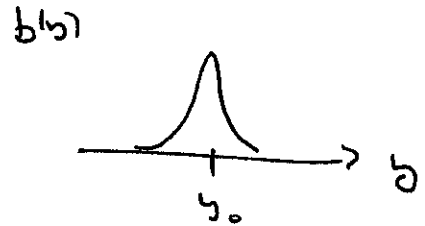
$$[(\Delta y)^2 + (\Delta x)^2]^{\frac{1}{2}} = (\Delta y) \left[ 1 + \left( \frac{dx}{dy} \right)^2 \right]^{\frac{1}{2}}$$

Then, the total time for the light beam to pass through the medium is

$$T = \int dy \frac{n(y)}{c} \left[ 1 + \left( \frac{dx}{dy} \right)^2 \right]^{\frac{1}{2}}$$

Now we need a formalism to minimize  $T$  with respect to variations of the path  $x(y)$ . Working with continuous functions, we can still vary  $x(y)$  by a localized perturbation, a bump  $b(y)$

$$x(y) \rightarrow x(y) + \alpha b(y)$$



If  $x(y)$  is the path of least time, we must find

$$\frac{dT}{d\alpha} = 0$$

at  $\alpha = 0$  after this substitution, for *any* bump-like perturbation  $b(y)$  that keeps the endpoints of the path fixed. Compute this derivative. Making the substitution

$$T(\alpha) = \int dy \frac{n(y)}{c} \left[ 1 + \left[ \frac{d}{dy} (x(y) + \alpha b(y)) \right]^2 \right]^{\frac{1}{2}}$$

Then

$$\frac{dT}{d\alpha} = \int dy \frac{n(y)}{c} \frac{\frac{dx}{dy} \cdot \frac{db}{dy}}{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{\frac{1}{2}}} + \mathcal{O}(\alpha)$$

To simplify this expression, integrate by parts. The path  $x(y)$  must start at A and end at B, so  $b(y) = 0$  at the initial and final values of  $y$ . This implies that the surface term from the integration by parts gives zero. Then

$$\frac{dT}{d\alpha} \Big|_{\alpha=0} = \int dy b(y) \left(-\frac{d}{dy}\right) \left[ \frac{n(y)}{c} \frac{dx/dy}{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{\frac{1}{2}}} \right]$$

If  $b(y)$  is a function that is steeply peaked near  $y = y_0$ , this integral can only vanish if

$$\frac{d}{dy} \left( n(y) \frac{dx/dy}{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{\frac{1}{2}}} \right) \Big|_{y=y_0} = 0$$

The function  $b(y)$  can be peaked anywhere, so this equation is satisfied at every point on the path. This is a differential equation that we can solve for  $x(y)$ .

The simplest case is that in which  $n(y)$  is a constant. In that case

$$\frac{n \frac{dx}{dy}}{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{\frac{1}{2}}} = \text{constant}$$

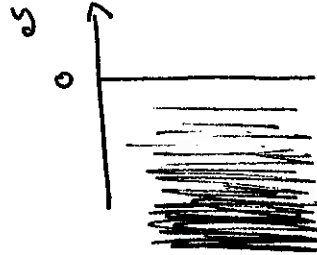
which implies

$$\frac{dx}{dy} = \text{constant}$$

That is,  $x(y)$  is a straight line, as we would have expected.

A more interesting case is an  $n(y)$  with a linear gradient,

$$n(y) = n_0 - \alpha y$$



We might find such a behavior in an estuary with fresh water on top and the concentration of salt in the water increasing systematically with depth. For this case

$$\frac{n(y) \frac{dx}{dy}}{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{\frac{1}{2}}} = C$$

that is

$$n^2(y) \left(\frac{dx}{dy}\right)^2 = C \left(1 + \left(\frac{dx}{dy}\right)^2\right)$$

which implies

$$\frac{dx}{dy} = \frac{C}{\left[n^2(y) - C\right]^{\frac{1}{2}}}$$

The integral of this equation is

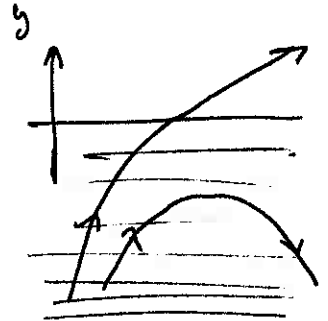
$$x(y) = \int^y \frac{C dy}{\left[(n_0 - \alpha y)^2 - C^2\right]^{\frac{1}{2}}}$$

which gives

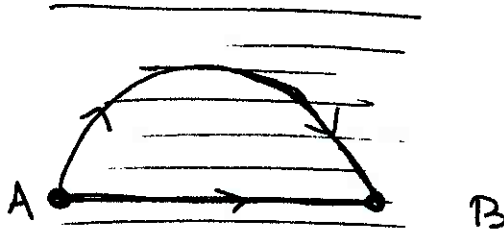
$$x = \frac{c}{a} \cosh^{-1} \left( \frac{n_0 - ay}{c} \right)$$

Finally,

$$y = \frac{n_0}{a} - \frac{c}{a} \cosh \frac{ax}{c}$$



Note that this formalism applies perfectly well even in situations with total internal reflection, where the light path reflects back down from the upper levels of the medium. In that case, a light source A deep in the medium and an observer B also deep in the medium might be joined by a path of least time. This formalism does not promise that the path is the absolute minimum of time, only that it is the local minimum among nearby paths.



We can streamline the mathematical analysis that led to the equation for the shortest time trajectory. Instead of varying  $x(y)$  by a bump function, we can consider an arbitrary variation

$$x(y) \rightarrow x(y) + \delta x(y)$$

Write the dependence of the total time  $T$  on the path as  $T[x(y)]$ . The square brackets denote a *functional* dependence, that is,  $T$  depends on the complete form of the

function  $x(y)$ . If  $T[x(y)]$  is minimized by a particular path  $x(y)$ , then if  $x(y)$  is perturbed in any way, the term of  $T$  linear in the perturbation must vanish. We can work out the explicit consequences of this statement by formally expanding  $T$  to linear order in  $\delta x(y)$ . For the time integral

$$T[x(y)] \rightarrow T[x+\delta x] = \int dy \frac{n(y)}{c} \left[ 1 + \left( \frac{d}{dy} (x + \delta x) \right)^2 \right]^{\frac{1}{2}}$$

Call the linear term  $\delta T$ :

$$\begin{aligned} \delta T &= \int dy \frac{n(y)}{c} \left[ 1 + \left( \frac{dx}{dy} \right)^2 + 2 \frac{dx}{dy} \frac{d\delta x}{dy} + \dots \right]^{\frac{1}{2}} - T \\ &= \int dy \frac{n(y)}{c} \frac{\frac{dx}{dy} \frac{d}{dy} (\delta x)}{\left[ 1 + \left( \frac{dx}{dy} \right)^2 \right]^{\frac{1}{2}}} \end{aligned}$$

Usually, we need to integrate by parts to write this expression as a simple integral with  $\delta x(y)$ . That is true in this case. We find

$$\delta T = - \int dy \delta x(y) \frac{d}{dy} \left( \frac{n(y)}{c} \frac{dx/dy}{\left[ 1 + (dx/dy)^2 \right]^{\frac{1}{2}}} \right)$$

Again, the surface term from integration by parts vanishes if

$$\delta x(y) = 0$$

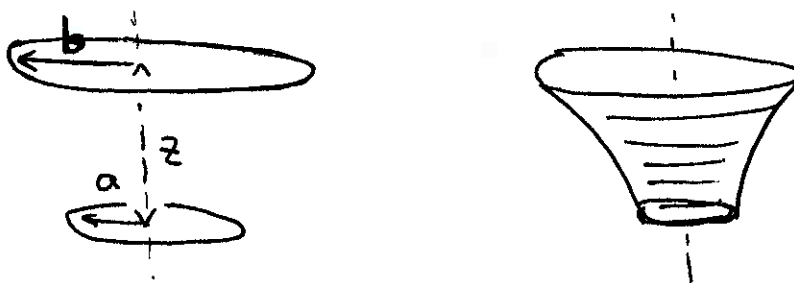
at the endpoints, which must be true if the beginning and ending points of the path are fixed. The variation  $\delta x$  is otherwise arbitrary, and so  $\delta T = 0$  for any variation only if

$$\frac{d}{dy} \left( \frac{n(y)}{c} \frac{dx/dy}{\left[ 1 + (dx/dy)^2 \right]^{\frac{1}{2}}} \right) = 0$$

This is the same equation that we found above. This equation expresses the minimization of the *functional*  $T[x]$  with respect to arbitrary functions  $x(y)$ . The method we have used is called *calculus of variations* or *functional calculus*.

I will now give two more examples of the application of the calculus of variations. Often, examples of the calculus of variations involve competition between two antagonistic principles. We will see such competition in these two cases.

The first example is a textbook mathematics problem: Given two circular rings of radii  $a$  and  $b$  placed parallel to one another with a separation  $Z$ ,  $a < b$ , find the surface of minimal area that has the rings as its boundary.

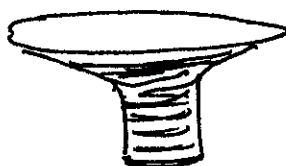


That is, find the shape of the film that forms on the system of rings when it is dipped into a soap solution. I will describe the surface as a surface of revolution  $r(z)$ .

There are two simple guesses about what this surface should be. The first one is that the surface should be a cone with linear variation of  $r$ . But this is not right, because we could make the surface smaller at the bottom without much penalty

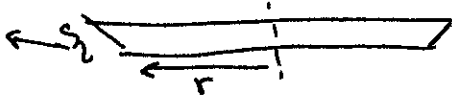


At the other extreme, we could keep  $r = a$  as long as possible and then transition to  $r = b$  at the last moment.



The surface of minimal area is the best compromise between these extreme positions.

The area of the surface is built up of rings

$$[(dz)^2 + (dr)^2]^{\frac{1}{2}}$$


of area

$$dA = 2\pi r [(dz)^2 + (dr)^2]^{\frac{1}{2}}$$

It is somewhat easier to analyze the problem if we think of  $z$  as a function of  $r$ , so that the area takes the form

$$A = \int_a^b dr \, 2\pi r \left[ 1 + \left( \frac{dz}{dr} \right)^2 \right]^{\frac{1}{2}}$$

To find the minimum of  $A$ , we vary

$$z(r) \rightarrow z(r) + \delta z(r)$$

and set  $\delta A = 0$

$$0 = \delta A = \int_a^b dr \, 2\pi r \frac{\frac{dz}{dr} \frac{d}{dr} \delta z}{\left[ 1 + \left( \frac{dz}{dr} \right)^2 \right]^{\frac{1}{2}}}$$

for an arbitrary variation  $\delta z(r)$ . Expose  $\delta z$  by integrating by parts

$$\delta A = 2\pi r \frac{\frac{dz}{dr}}{\left[ 1 + \left( \frac{dz}{dr} \right)^2 \right]^{\frac{1}{2}}} \delta z(r) \Big|_a^b - \int_a^b dr \, \delta z(r) \frac{d}{dr} \left[ 2\pi r \frac{\frac{dz}{dr}}{\left[ 1 + \left( \frac{dz}{dr} \right)^2 \right]^{\frac{1}{2}}} \right]$$

The boundary term vanishes because we must have  $\delta z(r) = 0$  at  $r = a$  and  $r = b$ , so that the surface joins neatly onto the rings. Then  $\delta A = 0$  only if the differential equation

$$\frac{d}{dr} \left( \frac{r \frac{dz}{dr}}{\left[1 + \left(\frac{dz}{dr}\right)^2\right]^{\frac{3}{2}}} \right) = 0$$

is satisfied.

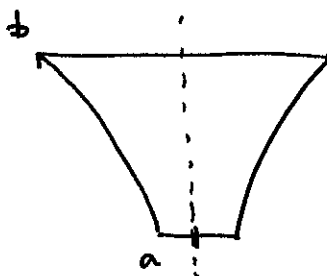
This differential equation is very similar to the one that we found in the previous problem, and we can find the solution in the same way. We find

$$z(r) = A \cosh^{-1} \frac{r}{A} + (\text{const.})$$

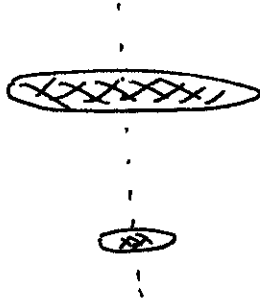
and, therefore

$$r = A \cosh \left( \frac{z - z_0}{A} \right)$$

The constants of integration  $A$ ,  $z_0$  can be eliminated in favor of  $a$  and  $b$ ,



As in the problem of the trajectory of a light ray, a solution of this differential equation is not necessarily the surface that gives the global minimum of the area. For  $Z$  large enough, the soap film forms as follows:



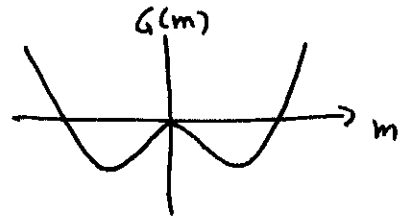
For a given  $a$ ,  $Z$ , the maximum value of  $b$  that can be accommodated by the solution above is

$$b = a \cosh \frac{Z}{a}$$

For values of  $b$  below this value, the solution with horizontal soap films can still be the global minimum.

The second example is drawn from statistical mechanics. Consider a material with two coexisting thermodynamic phases, for example, liquid and gas or magnetization up and down. The local thermodynamic state can be described by a variable  $m$  representing the magnetization, density, or whatever is appropriate. The free energy should be a function of  $m$  with two degenerate minima, representing the two phases. For simplicity, I will represent the free energy as a symmetrical function of  $m$ ,

$$G = -\frac{1}{2}Am^2 + \frac{1}{4}Bm^4$$



This function is minimized at

$$m^2 = \frac{A}{B} \quad \text{or} \quad m = \pm \left(\frac{A}{B}\right)^{\frac{1}{2}}$$

representing the fact that the system can exist stably with  $m$  taking either of these values.

We can extend this description to build a theory of the interface between phases



This is a boundary at which  $m$  is a function of position  $m(z)$  that goes continuously from one minimum to the other. We can model the variation of  $m$  by generalizing the free energy to a *functional*

$$G[m] = \int dz \left\{ \frac{1}{2} \rho \left( \frac{dm}{dz} \right)^2 - \frac{1}{2} A m^2(z) + \frac{1}{4} B m^4(z) \right\}$$

I assign a penalty to a state with a rapid variation of  $m$ , which is not a stable thermodynamic situation. The form of the interface is then given by minimizing  $F[m]$ . There is clearly a competition between keeping  $m(z)$  as close as possible to one of the minima  $\pm(A/B)^{1/2}$  and allowing  $m(z)$  to make the transition from one minimum to the other smoothly. We can resolve this competition using the calculus of variations.

To do this, compute the variation of  $G[m]$ ,

$$\delta G = \int dz \left\{ \rho \frac{dm}{dz} \frac{d}{dz} \delta m - A m \delta m + B m^3 \delta m \right\}$$

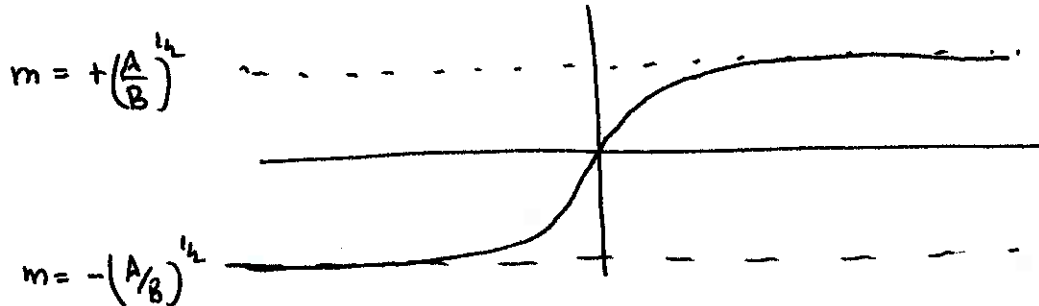
To group the factors  $\delta m$  together, integrate by parts

$$\delta G = \int dz \delta m(z) \left\{ -\rho \frac{d^2}{dz^2} m - A m + B m^3 \right\}$$

The boundary term vanishes because  $m(z)$  must be very close to one of the two minima as  $z \rightarrow \infty$  or  $z \rightarrow -\infty$ . Then the condition for the minimum is

$$\rho \frac{d^2 m}{dz^2} = -A m + B m^3$$

We want to find a solution to this equation of the general form



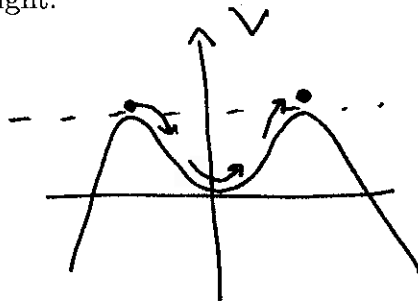
The variational equation looks like an equation from particle mechanics. If we replace  $z$  by  $t$  and  $m$  by the particle position  $x$ , it is exactly Newton's law for the force

$$-Am + Bm^3$$

The energy integral is

$$E = T + V = \frac{1}{2} \rho \left( \frac{dm}{dt} \right)^2 + \frac{1}{2} Am^2 - \frac{1}{4} Bm^4$$

Notice that the potential energy is the negative of  $Q(m)$ . This potential is unstable, with maxima at the points  $m = \pm(A/B)^{1/2}$ . A motion of the form that we are looking for is obtained in the following way: The particle starts at top of the hill on the left with velocity essentially zero. After a long time, the particle rolls off the hill and through the valley. As  $t$  or  $z \rightarrow \infty$ , the particle comes to rest at the top of the hill on the right.



For this solution, the energy is

$$E = \left[ \frac{1}{2} A m^2 - \frac{1}{4} B m^4 \right] \Big|_{m^2 = A/B} = + \frac{1}{4} \frac{A^2}{B}$$

Then

$$\frac{1}{2} \rho \left( \frac{dm}{dz} \right)^2 = \frac{1}{4} B \left( m^2 - \frac{A}{B} \right)^2$$

If  $dm/dz > 0$ ,  $|m| < (A/B)^{1/2}$ ,

$$\left[ \frac{2\rho}{B} \right]^{\frac{1}{2}} \frac{dm}{dz} = \left( \frac{A}{B} - m^2 \right)$$

This is easily integrated,

$$\int \frac{dm}{\sqrt{A/B - m^2}} = \left[ \frac{B}{2\rho} \right]^{\frac{1}{2}} \int dz$$

$$\left( \frac{B}{A} \right)^{\frac{1}{2}} \tanh^{-1} \left[ m \left( \frac{B}{A} \right)^{\frac{1}{2}} \right] = \left[ \frac{B}{2\rho} \right]^{\frac{1}{2}} (z - z_0)$$

Then

$$m(z) = \left( \frac{A}{B} \right)^{\frac{1}{2}} \tanh \left( \frac{A}{2\rho} \right)^{\frac{1}{2}} (z - z_0)$$

is the shape of the interface that gives minimum free energy and thus is the shape that is will be preferred thermodynamically. Notice that the position of the interface

is not fixed, so this position  $z_0$  remains as a free parameter in the solution. This freedom reflects the translation invariance of the original problem.

These examples, and particularly the last one, give us clues toward the formulation of a variational principle for particle mechanics.