Nov. 16

The Period-Doubling Route to Chaos

At the end of the previous lecture, we studied the transition to chaos in a simple model of ordinary differential equations. In that model, we saw, the chaotic state is the result of an instability of a stationary state of the model. There are many examples in which chaos appears in a mechanical system as the result of one or a finite number of instabilities. However, there are also examples in which the chaotic state is the result of an accumulation of structure involving an infinite number of successive instabilities. This is the *period-doubling route to chaos*, discovered by Mitchell Feigenbaum. In this lecture, I will study a very simple system that illustrates this scenario.

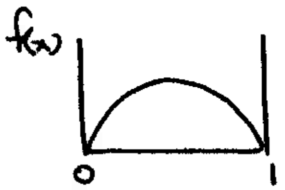
Feigenbaum studied a system that is one level down from a differential equation—a simple recursion

$$x_{n+1} = f(x_n)$$

In this lecture, I will concentrate on the example of the *logistic equation*, in which

$$f(x) = 4\lambda x(1-x)$$

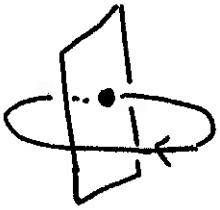
$$0 < x < 1$$



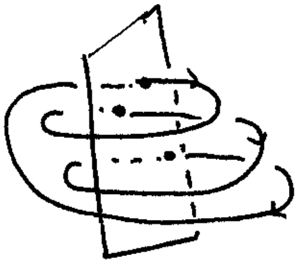
The parameter λ will be the control parameter. For small values of λ, the recursion will have a fixed point. At higher values, the behavior of the recursion will become increasingly complex and, eventually, we will find a chaotic state.

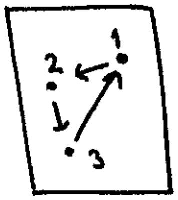
It is not so obvious that such a simple model has anything in common with the continuous time evolution in mechanics. However, you might imagine that a recursion equation models the successive intersections of a trajectory with the Poincaré section. Let λ be a control parameter for the mechanics problem. Then, for small λ we might have a periodic orbit. This would correspond to a fixed point of the recursion.

$$x_* = f(x_*)$$



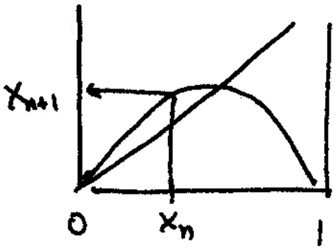
For some higher value of λ, the periodic orbit will become unstable, and we will find a pattern of multiple intersections realized in a specific order.



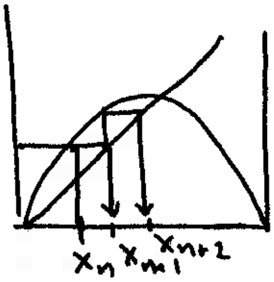


This is the dynamics modelled by the recursion formula.

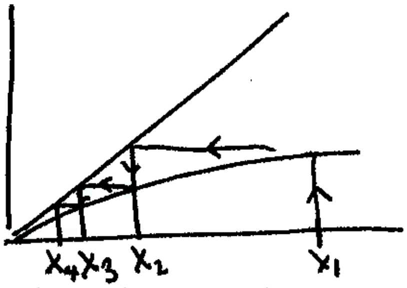
To analyze the logistic equation, it is useful to visualize the interation graphically. Draw the function $y = f(x)$ together with the identity $y = x$. Given x_n, we can find x_{n+1} from the graph.



Even better, we can reflect this value down to the x axis and use it to produce the following value x_{n+2}.



For small values of λ, $\lambda < \frac{1}{4}$, this process looks like



From any initial starting point, $x_n \to 0$ as $n \to \infty$. By linearizing the recursion,

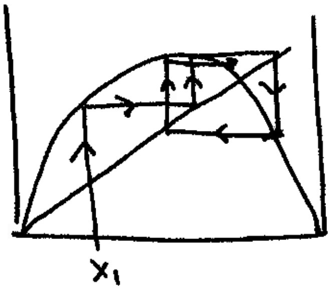
$$x_{n+1} \cong 4\lambda x_n$$

we see that, if x_N is in the region where linearization is accurate, then

$$x_{N+k} \cong (4\lambda)^k x_N$$

So the fixed point $x = 0$ is stable for $\lambda < \frac{1}{4}$. For $\lambda > \frac{1}{4}$, this point is unstable and we need to look for a new stable fixed point.

For $\lambda > \frac{1}{4}$, the logistic function $f(x)$ has the form



and the recursion takes us to the nonzero intersection of this curve with the diagonal. This is a fixed point, located at the solution of the equation

$$x_* = 4\lambda x_* (1 - x_*)$$

or

$$1 = 4\lambda (1 - x_*)$$

That is,

$$x_* = 1 - \frac{1}{4\lambda}$$

We should next compute the stability of this point. Let Δx_n be a small deviation from the fixed point

$$x_n = x_* + \Delta x_n$$

Then

$$x_{n+1} = x_* + \Delta x_{n+1} = 4\lambda (x_* + \Delta x_n)(1 - x_* - \Delta x_n)$$

Keeping only terms up to those linear in Δx,

$$x_* + \Delta x_{n+1} = 4\lambda\, x_*(1-x_*) + \Delta x_n \; 4\lambda \; (1 - x_* - x_*)$$

$$\Delta x_{n+1} = \Delta x_n \cdot 4\lambda \cdot (1 - 2x_*)$$

so that

$$\Delta x_{n+1} = c \;\; \Delta x_n$$

where the constant c is given by

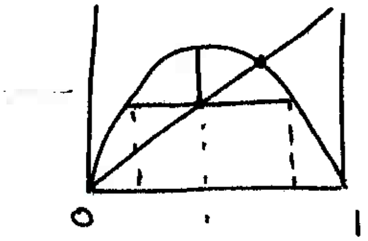
$$c = 4\lambda(1 - 2x_*) = (2 - 4\lambda)$$

At $\lambda = \frac{1}{4}$, $c = 1$. For larger values of λ, $c < 1$, so that deviations from the fixed point die out with successive interations. However, at $\lambda = \frac{3}{4}$, c reaches the value (-1). Beyond this point $|c| > 1$ and the fixed point is unstable.

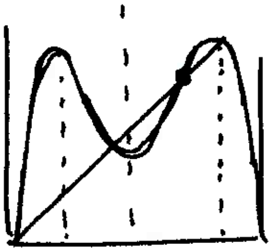
What happens next? To analyze this, consider the interated transformation

$$f^{(2)}(x) = f(f(x))$$

We can work out the form of the function $f^{(2)}(x)$ graphically. There are two maxima, the two image points of the maximum of $f(x)$ at $x = \frac{1}{2}$.



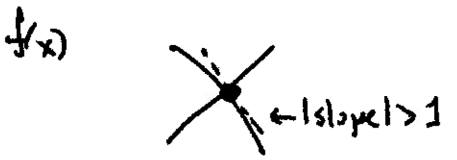
The fixed point x_* of $f(x)$ must also be a fixed point of $f^{(2)}(x)$. Then $f^{(2)}(x)$ has the form

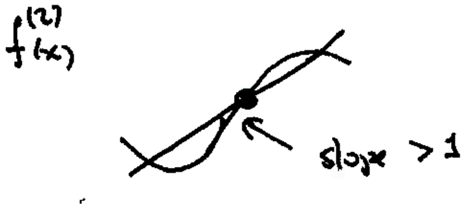


In particular, since the linearizations of $f(x)$ and $f^{(2)}(x)$ are related by

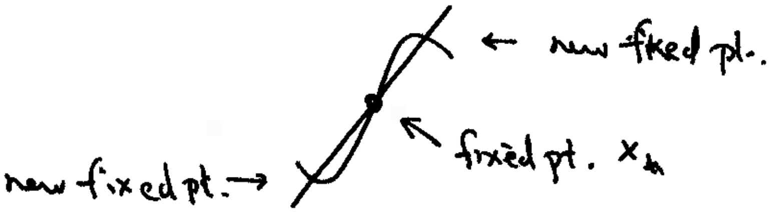
$$\Delta X_{n+1} = C\, \Delta X_n \quad \Rightarrow \quad \Delta X_{n+2} = C^2\, \Delta X_n$$

when $c < -1$ and the slope of $f(x)$ at x_* is greater than 1, the slope of $f^{(2)}(x)$ at x_* is also greater than 1.

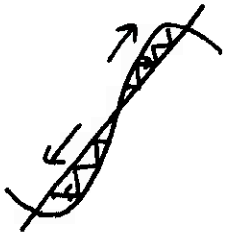




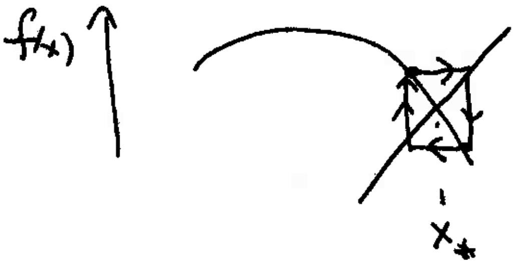
Then the form of $f^{(2)}(x)$ near the fixed point is



In this structure, the recursion is

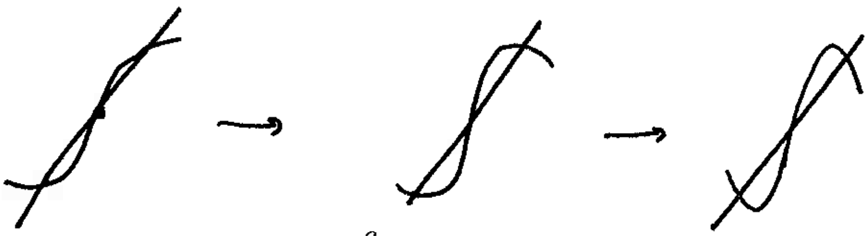


so that the two new fixed points are stable. These two new fixed points are not fixed points of $f(x)$, so it must be that f(x) takes one into the other and vice versa.



This is a stable cycle of $f(x)$ of period 2.

This period 2 cycle is stable for some region of λ above $\lambda = \frac{3}{4}$. But as λ increases, the form of the function $f^{(2)}(x)$ evolves

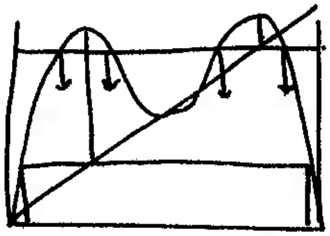


and the slope at the new fixed points of $f^{(2)}(x)$ reaches (-1). At this point, the period 2 cycle becomes unstable. This transition occurs at $\lambda_2 = 0.8623$.

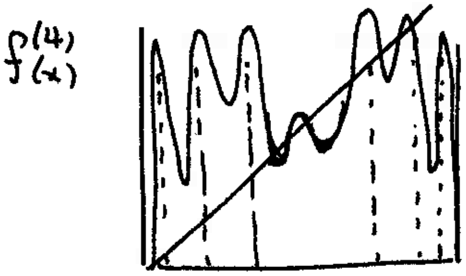
We can analyze this instability by constructing the iterate $f^{(4)}(x)$,

$$f^{(4)}(x) = f^{(2)}(f^{(2)}(x)) = f(f(f(f(x))))$$

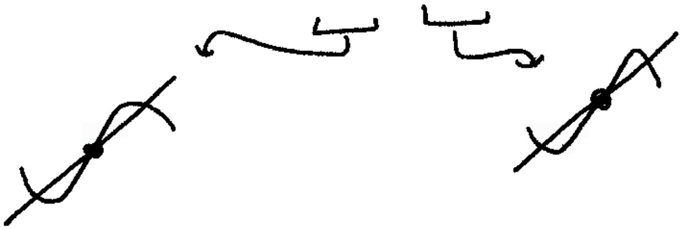
The function $f^{(4)}(x)$ has maxima at the image points of the two maxima of $f^{(2)}(x)$. There are 6 of these in all.



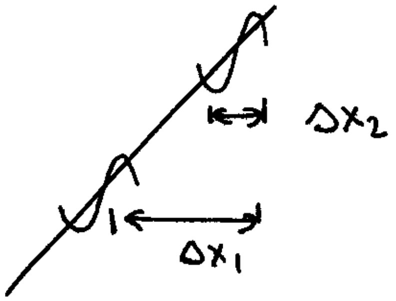
The three fixed points of $f^{(2)}(x)$ remain fixed points of $f^{(4)}(x)$. Then $f^{(4)}(x)$ must have the form



The regions near the two points forming the period 2 cycle of $f(x)$ have the form



The pattern is the same as we saw in $f^{(2)}(x)$. The fixed points each generate a pair of new fixed points that are stable in the region of λ just above the value where the previous fixed points go unstable. We now have 4 stable fixed points, forming a stable cycle of the original iteration of period 4. These pieces of $f^{(4)}(x)$ are very similar in form to the corresponding piece of $f^{(2)}(x)$, except for a change in scale



We can write

$$\frac{\Delta x_1}{\Delta x_2} = \alpha_2$$

Again, as λ increases, the slope of the iterate at the new stable fixed points increases until it reaches (-1). For $f^{(4)}(x)$, this happens at $\lambda_3 = 0.885$. Then the pattern repeats once again. The cycle of period 4 becomes unstable with respect to a stable cycle of period 8. As λ continues to increase, each successive cycle becomes unstable. We find, successively, stable cycles of period 16, 32, 64, etc. I will denote the value of λ at which the cycle of period 2^n becomes stable as λ_n.

For the first three values of λ_n, we found

$$\frac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_2} = 4.934$$

Feigenbaum observed that this ratio tends to a constant

$$\frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n} \xrightarrow[n \to \infty]{} \delta = 4.6692..$$

At the same time, the amplitude of the new component decreases systematically, and the ratio of successive amplitudes also approaches a limit

$$\frac{\Delta x_n}{\Delta x_{n+1}} \longrightarrow \alpha = 2.5029$$

Note that the successive transition points λ_n are approximated by

$$\lambda_{n+1} - \lambda_n \cong \frac{1}{\delta}\left(\lambda_n - \lambda_{n-1}\right)$$

$$\lambda_{n+1} \approx (\text{const}) + \frac{c}{\delta} + \frac{c}{\delta^2} + \cdots \frac{c}{\delta^n}$$

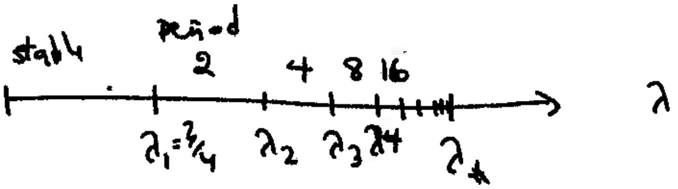
or

$$\lambda_{n+1} \cong A + \frac{c}{\delta}\,\frac{1 - 1/\delta^n}{1 - 1/\delta} \cong A + B\left(1 - \frac{1}{\delta^n}\right)$$

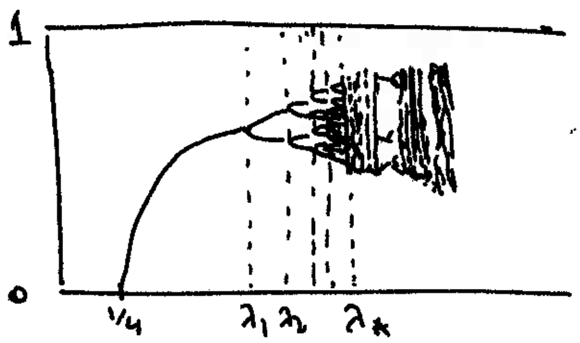
Then these points accumulate at an asymptote λ_*, given approximately by

$$\lambda_* \approx \lambda_{n+1} + \frac{B}{\delta^n}$$

Thus, the succession of period-doubling transitions goes all the way to infinite period at a finite value $\lambda = \lambda_* = 0.8925$.



Beyond λ_*, arbitrary periods, not of the form 2^n, are possible. In portions of the region $\lambda > \lambda_*$, the set of points generated by the recursion is dense in a subinterval of [0,1]. Then the recursion has all of the properties of chaos: sensitivity to initial conditions, a space-filling trajectory, and orbits of arbitrary period. However, there are also regions of λ beyond λ_* where the trajectories are simple, with periods 3 and 7 appearing in some regions. The global behavior of the recursion can be visualized by plotting, for each value of λ the points associated with a stable orbit. This diagram has the form



The presence of orbits with any period is not so obvious, but there are two cases it which this is readily demonstrated. First, at $\lambda = 1$, it is very easy to construct periodic orbits of all periods. At this λ, the recursion is

$$x_{n+1} = 4x_n(1-x_n)$$

Let

$$x_n = \tfrac{1}{2}(1-\cos 2\pi y_n)$$

Note that the values of y_n from 0 to $\frac{1}{2}$ cover the values of x_n from 0 to 1. Plugging this into the recursion

$$\begin{aligned} 4x_n(1-x_n) &= 4\cdot\tfrac{1}{4}(1-\cos 2\pi y_n)(1+\cos 2\pi y_n) \\ &= \sin^2 2\pi y_n \\ &= \tfrac{1}{2}(1-\cos 4\pi y_n) \end{aligned}$$

So it would seem that the recursion is

$$y_{n+1} = 2y_n$$

However, for $y_n > \frac{1}{4}$, we must reflect y_{n+1} back into the interval $[0, \frac{1}{2}]$ by choosing another y that gives the same cosine. The recursion is then, more properly,

$$y_{n+1} = \begin{cases} 2y_n & 0 < y_n < \frac{1}{4} \\ 1 - 2y_n & \frac{1}{4} < y_n < \frac{1}{2} \end{cases}$$

Then, if $y_0 = \frac{1}{3}$, we find the sequence

$$\frac{1}{3} \longrightarrow \frac{1}{3} \longrightarrow \frac{1}{3}$$

Similarly

$$\frac{1}{5} \rightarrow \frac{2}{5} \rightarrow \frac{1}{5} \rightarrow \frac{2}{5} \rightarrow$$

$$\frac{1}{9} \rightarrow \frac{2}{9} \rightarrow \frac{4}{9} \rightarrow \frac{1}{9} \rightarrow \frac{2}{9} \rightarrow$$

and, in general, the starting point

$$\frac{1}{2^n + 1}$$

gives a cycle of period n.

Near $\lambda = 0.96$ there is a region with a stable orbit of period 3. I will now show that, if a recursion has a periodic orbit with period 3, it also has periodic orbits with all other periods. This result was presented by Li and Yorke in a paper entitled "Period 3 Implies Chaos", That is true, however, only if the period 3 orbit is unstable. In any case, here is the proof.

I consider a continuous mapping

$$x_{n+1} = f(x_n)$$

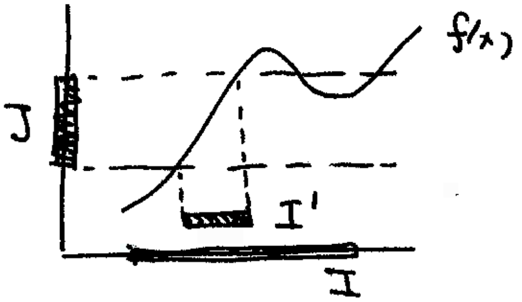
that generates a recursion. I assume that there is a cycle of period 3

$$a \longrightarrow b \longrightarrow c \longrightarrow a$$

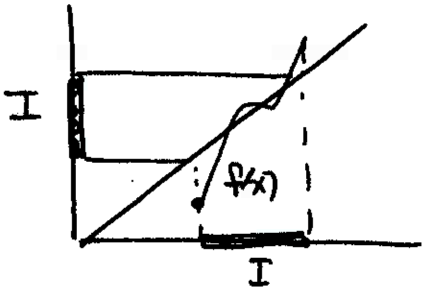
with $a < b < c$. We can always choose a to be the smallest of the three values, but there is a distinct case $a < c < b$. However, for that case, there is a parallel argument.

If $f(x)$ is a continuous function and $f(a) = b$, $f(b) = c$, then all points in the interval (b, c) must be values of $f(x)$ for some values of x between a and b. We can use this observation in conjunction with properties of maps of intervals:

First, if $I = (x_1, x_2)$ and $J = (x_3, x_4)$ are intervals and $J \subset f(I)$, then there is a subinterval I' of I such that $J = f(I')$.



Second, if $I \subset f(I)$, then there is a fixed point of $f(x)$ in I. This follows because the graph of $f(x)$ must cross the diagonal in I.



Let $I_0 = (a, b)$, $I_1 = (b, c)$. Then, as I argued above,

$$I_1 \subset f(I_0)$$

Also, since $f(b) = c$, $f(c) = a$, the whole interval (a, c) must be in the image of I_1. Thus

$$I_0 \cup I_1 \subset f(I_1)$$

Now we can show that there are cycles of the recursion with any period. For period 1, note that

$$I_1 \subset f(I_1)$$

This implies that there is a fixed point of $f(x)$ in I_1.

For period 2, note that

$$I_0 \subset f(I_1) \subset f(f(I_0))$$

So, there is a fixed point of $f^{(2)}(x)$ in I_0. Since the image of any point in I_0 is in the disjoint interval I_1, this must correspond to a cycle of period 2 of $f(x)$ with one point in I_0 and one point in I_1.

For period 3, note that, since $I_1 \subset f(I_1)$, there is a subinterval $A_1 \subset I_1$ such that $I_1 = f(A_1)$. Since $A_1 \subset f(I_0)$, there is a subinterval $B \subset I_0$ such that $A_1 = f(B)$. Then we have

$$B \xrightarrow{f} A_1 \xrightarrow{f} I_1 \xrightarrow{f} B$$

and, since

$$B \subset f(f(f(B)))$$

the interate $f^{(3)}(x)$ must have a fixed point in B. This point gives a period 3 cycle of $f(x)$ with one point in I_0 and the other points in the disjoint interval I_1.

For period 4, use the intervals above, and note that there is an interval $A_2 \subset A_1$ such that $A_1 = f(A_2)$. Then let B be such that $A_2 = f(B)$. We now have mappings

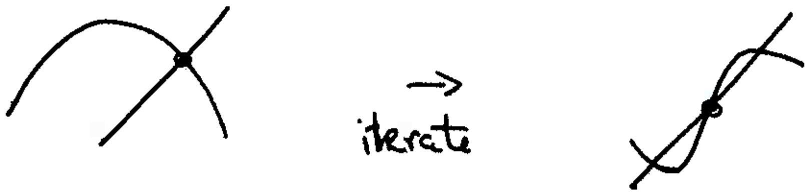
$$B \xrightarrow{f} A_2 \xrightarrow{f} A_1 \xrightarrow{f} I_1 \xrightarrow{f} B$$

that imply that there exists a fixed point of $f^{(4)}(x)$ in B. This point gives a cycle of period 4 of $f(x)$.

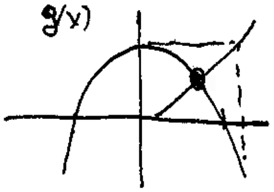
Continuing in this way, adding A_3 such that $A_2 = f(A_3)$ gives a cycle of period 5, adding A_4 such that $A_3 = f(A_4)$ gives a cycle of period 6, and so forth. These trajectories of higher period may be stable or unstable with respect to the trajectories of lower period. When the trajectories of low period are state, we have regular motion. When the trajectories of all finite period are unstable, we have chaos.

Return now to the discussion of the transition to chaos. So far in this lecture, I have only discussed the case of a recursion with the logistic function. Our discussion seemed to be very special to that system. However, Feigenbaum also explored a number of other functions for the recursion and found the same pattern of period-doubling transitions, a different values of the control parameter but with the same values of δ and α. These systems are all described by a general theory.

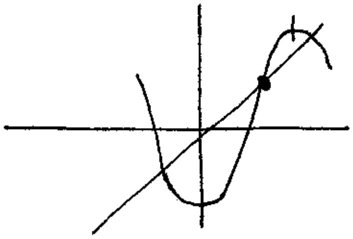
In our discussion of the logistic system, we saw that the motif



continually repeated at each period-doubling transition. In fact, these shapes approach a universal form. To define the problem better, shift and rescale $f(x)$ to a function $g(x)$ such that $g(x)$ has its maximum at $x = 0$ and $g(0) = 1$,



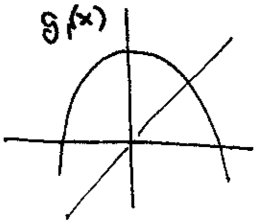
The function $g(x)$ has a fixed point between $x = 0$ and $x = 1$. Above the transition to the next period-doubling, the slope at this point will less than (-1). Then $x = 0$ will be a local *minimum* of $g^{(2)}$(x), and the iterate will contain a smaller-scale structure with the unstable fixed point in the middle.

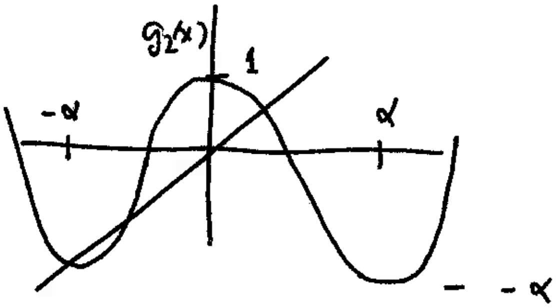


Similarly, above the transition to the following period-doubling, the slope at the fixed point nearest to $x = 0$ will be less than (-1), and the iterate will contain still smaller-scale structure. We can attempt to stabilize the form of $g(x)$ by iterating with a change of scale

$$g_{n+1}(x) = -\alpha\, g_n(g_n(x/\alpha))$$

so that $g_{n+1}(x)$ derived from $g_n(x)$ also gives an extremum (now a maximum) at $x = 0$ with $g_{n+1}(0) = 1$,





It is especially interesting to look for a function that is *invariant* under the transformation just described for an appropriate value of α,

$$g_*(x) = -\alpha \; g_*(g_*(x/\alpha))$$

This function would a fixed point in the space of function interations. It would describe the limiting behavior of period-doubling transitions, the behavior exactly at $\lambda = \lambda_*$. Feigenbaum gave a method for constructing this function and found the appropriate value of the rescaling,

$$\alpha = 2.5029$$

He could then study the behavior of small perturbations of this function under iteration. He found that there is one unstable direction

$$g_n(x) = g_*(x) + \delta^n h(x)$$

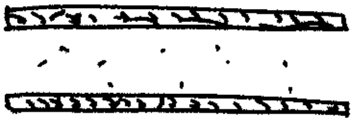
The eigenvalue δ can be interpreted as the ratio of deviations corresponding to successive transitions, and indeed,

$$\delta = 4.669$$

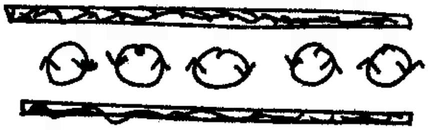
All other perturbations are stable under the iteration. Thus, the function $g_*(x)$ is an *attractor* in the space of possible recursions, up to the necessity of adjusting one control parameter to reach the limit λ_* of period-doubling bifurcations. Any interation sufficiently close to $g_*(x)$ will be drawn into the attraction and give a period-doubling cascade with the same, *universal* values of α and δ.

Feigenbaum showed that these values of α and δ apply to other simple function interations. However, it is possible to draw a much more interesting conclusion by going back to our original point of view in which these function iterations are models of the successive intersections of the continuous trajectory of a mechanical system with the Poincaré section. If the iteration that actually describes a mechanical system lies in the domain of attraction of Feigenbaum's fixed point, then the mechanical system should also show a period-doubling cascade of transitions with the same universal scaling behavior.

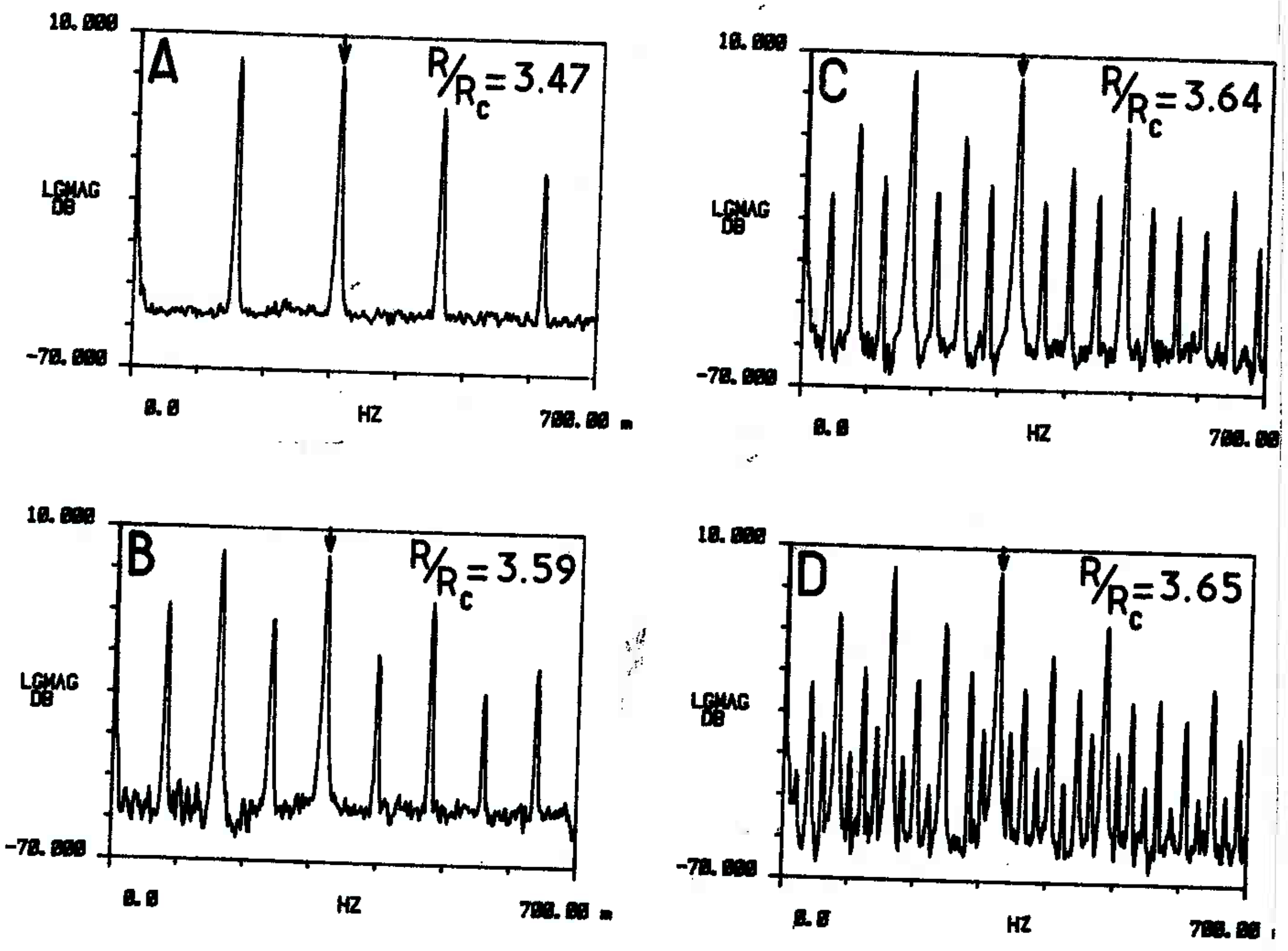
Indeed, there is evidence for such period-doubling cascades in many nonlinear systems described by ordinary differential equations, including nonlinear oscillators and even a model of the heart. The most remarkable example comes in a fluid-mechanical system, the Rayleigh-Bénard instability, studied by Libchaber and collaborators. Consider a fluid at rest between two horizontal slabs, each held at fixed temperature.



Heat the bottom, cool the top. The control parameter is the *Rayleigh number R*, proportional to the temperature difference between the plates. At some large enough temperature difference, the system will be unstable with respect to convection, a set of flows of the form



The instability to convection turns out to be part of a series of instabilitites to fluid motions of increasing complexity. This is easiest to study in a medium in which the thermal conductivity is much greater than the viscosity. Libchaber chose liquid mercury between copper plates, in a magnetic field. The first instability is to a pattern of rolls, as shown above. At a higher temperature gradient, there is a further instability to a time-dependent flow with one dominant frequency. As the temperature gradient is increased, there is a series of period-doubling transitions. In Libchaber, Larouche, and Fauve, J. de Phys. 43, L211 (1982), this evolution is followed up through the transition to period $16 = 2^4$ relative to the original period. Fourier



A
R/R_c=3.47
10.000
LGMAG DB
-70.000
0.0
HZ
700.00 m
C
R/R_c=3.64
10.000
LGMAG DB
-70.000
0.0
HZ
700.00
B
R/R_c=3.59
10.000
LGMAG DB
-70.000
0.0
HZ
700.00 m
D
R/R_c=3.65
10.000
LGMAG DB
-70.000
0.0
HZ
700.00

spectra of a local temperature reading are shown in the figure. The Rayleigh numbers of the last three transitions obey

$$\frac{R_3 - R_2}{R_4 - R_3} = 4.4 \pm 0.1$$

in reasonable agreement with the value of $\delta = 4.669$ predicted by Feigenbaum's theory.