Oct. 21

Small Oscillations and Stability

In this course, we have seen a number of examples in which a dynamical system can be at rest at an equilibrium point. We have seen examples in which this point is stable with respect to small disturbances and other examples in which the equilibrium is unstable. In this lecture, I will explain how one analyzes the nature of an equilibrium systematically by expanding the nonlinear equations of motion about the equilibrium point. In later lectures, we will discuss systems in which the equilibrium is a periodic motion and problems arise that need more sophisticated methods. This analysis, though, is the starting point.

Begin with the simplest example, a particle moving in a 1-dimension potential. The equation of motion is

$$\ddot{x} = f(x)$$

Let x_0 be a point for which $f(x_0) = 0$. This is an *equilibrium point.* The initial conditions

$$x = x_0 \quad , \quad \dot{x} = 0$$

yield the solution

$$x = x_0$$

for all later times.

Now expand about this solution. Write

$$x = x_0 + \xi(t)$$

and regard $\xi(t)$ as small, so that we can drop terms quadratic and higher in ξ. Then $\xi(t)$ obeys the equation

$$\ddot{\xi} = f'(x_0)\,\xi + \mathcal{O}(\xi^2)$$

If

$$f'(x_0) < 0$$

then there is a restoring force that counteracts the displacement of ξ. In that case, write

$$f'(x_0) = -\omega^2$$

then this equation becomes

$$\ddot{\xi} = -\omega^2 \xi + \cdots$$

The general solution to this equation is

$$\xi = A\cos\omega t + B\sin\omega t$$

We call this a *stable oscillation.* As we saw in the first lecture, such a motion is not *structurally stable* in the mathematicians' sense, because a small nonlinear term can lead to growth or decay of the amplitude of the oscillation. In practice, though, most systems with stable oscillation are slightly damped by interactions that extract energy from the oscillation, so this kind of behavior typically leads to motion that stays within the vicinity of the equilibrium point.

The alternative case is

$$f'(x_0) > 0$$

In this case, we can write

$$f'(x_0) = +\lambda^2$$

The ξ equation becomes

$$\ddot{\xi} = \lambda^2 \xi + \mathcal{O}(\xi^2)$$

The general solution to this equation is

$$\xi = A e^{\lambda t} + B e^{-\lambda t}$$

There is a special trajectory, corresponding to $A = 0$, that approaches the equilibrium point. However, every other trajectory eventually runs off to large distances from the equilibrium point, with the distance increasing exponentially with time. When ξ becomes sufficiently large, the approximation that we are in the neighborhood of the equilibrium breaks down. This behavior is called a *linear instability.*

Similar behaviors are seen in systems with several degrees of freedom. For simplicity, I will ignore possible velocity-dependent damping forces and write the equations of motion as

$$\ddot{q}_i = f_i(q)$$

where the q_i are (generalized) coordinates. The forces $f_i(q)$ might in principle depend on all of the q_i. Let $\{q_i^0\}$ be a choice of the q_i for which

$$f_i(q^0) = 0$$

This is an equilibrium configuration of the system, one for which

$$q_i = q_i^0$$

is a solution for all time. To work out the stability of this equilibrium, we linearize about the equilibrium point. Write

$$q_i = q_i^0 + \eta_i(t)$$

The small deviations $\eta_i(t)$ obey the equation

$$\ddot{\eta}_i = \left.\frac{\partial f_i}{\partial q_j}\right|_{q=q_0} \cdot \eta_j + O(\eta^2)$$

It is natural to write this as a *matrix* equation. Define

$$\eta = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix} \qquad A = \begin{pmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} & \cdots & \frac{\partial f_1}{\partial q_n} \\ \vdots & & & \\ \frac{\partial f_n}{\partial q_1} & \cdots & & \end{pmatrix}$$

Then the equation is

$$\ddot{\eta} = A\eta$$

To find the behavior of the solutions, we diagonalize the matrix A. In general, A is a real-valued matrix that is not necessarily symmetric. However, in general, such an $n \times n$ matrix still has a complete set of eigenvectors. More precisely, it has a complete set of *right eigenvectors* ζ_a, $a = 1, \ldots, n$, satisfying

$$A\zeta_a = \lambda_a^2 \zeta_a$$

and a complete set of *left eigenvectors* θ_a satisfying

$$\theta_a A = \lambda_a^2 \theta_a$$

The eigenvalues are the same, since

$$\theta_a A \zeta_b = \lambda_a^2 (\theta_a \zeta_b) = (\theta_a \zeta_b) \lambda_b^2$$

and this also implies

$$\theta_a \zeta_b = 0 \quad \text{for} \quad \lambda_a^2 \neq \lambda_b^2$$

One special case is very simple: For a collection of particles with positions x_i and masses m_i moving in a potential $V(x_1, \ldots, x_n)$, the equations of motion are

$$m_i \ddot{x}_i = -\frac{\partial V}{\partial x_i}$$

An equilibrium point is a point such that

$$-\frac{\partial V}{\partial x_i}\bigg|_{x_i = x_i^0} = 0$$

Let $x_i = x_i^0$ be an equilibrium point, and expand about this point by writing

$$x_i = x_i^0 + \xi_i(t)$$

Then the ξ_i obey

$$m_i \ddot{\xi}_i = -\left[\frac{\partial^2 V}{\partial x_i \partial x_j}\right]_{x=x^0} \cdot \xi_j$$

or

$$\ddot{\xi}_i = -\frac{1}{m_i}\left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)_{x^0} \cdot \xi_j$$

The matrix A in the more general analysis above is not symmetric, but if we redefine

$$\eta_i = \sqrt{m_i}\,\xi_i$$

these new variables obey

$$\ddot{\eta}_i = \left(-\frac{1}{\sqrt{m_i}} \frac{\partial^2 V}{\partial x_i \partial x_j}(x^0) \frac{1}{\sqrt{m_j}} \right) \eta_j$$

or

$$\ddot{\eta}_i = B_{ij} \eta_j$$

with a real symmetric matrix B given by

$$B_{ij} = -\frac{1}{\sqrt{m_i}} \frac{\partial^2 V}{\partial x_i \partial x_j}(x') \frac{1}{\sqrt{m_j}}$$

This can be diagonalized in the standard way. If μ_a are the eigenvectors of B, the right and left eigenvectors of A are given by

$$(\eta_a)_i = \frac{1}{\sqrt{m_i}} (\mu_a)_i \qquad (\theta_a)_i = (\mu_a)_i \sqrt{m_i}$$

In any event, we can analyze A and find its right eigenvectors, also called *normal modes*, and the corresponding eigenvalues. We can then write the general solution to the η equation as

$$\eta(t) = \sum_a \eta_a \left[a_a e^{\lambda_a t} + b_a e^{-\lambda_a t} \right]$$

where λ_a, $-\lambda_a$ are the two square roots of the eigenvalue λ_a^2. For each eigenvalue, there are three possible cases:

1. λ_a^2 is a real negative number. Then

$$\lambda_{a,-} = \pm i\omega_a \qquad \omega_a \text{ real}$$

This leads to a stable oscillation in the direction of the eigenvector ζ_a.

2. λ_a^2 is a real positive number. Then

$$\lambda_{a,-} = \pm \lambda_a \qquad \lambda_a \text{ real}$$

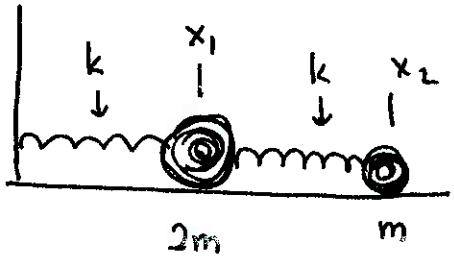
This leads to a linear instability in the direction of the eigenvector ζ_a.

3. λ_a^2 is a complex number. Then the square root of λ_a^2 will have a nonzero real part. This will be positive for one of λ_a, $-\lambda_a$ and will lead to a linear instability in the direction of ξ_a. Note that, in the example above of massive particles interacting with a potential, the eigenvalues of B are real and so this case does not occur.

If all of the eigenvalues λ_a^2 are negative real numbers, the solution for $\eta(t)$ above will only include oscillations and the system will be stable. However, if any of the λ_a^2 are positive, the system will run away from the equilibrium exponentially along the direction of the corresponding eigenvector.

I will now illustrate this general formalism with several examples. First, I will study some systems for which the equations of motion are linear and the solutions contain only stable oscillations. Such systems occur often in mechanics and in atomic physics, and it is important to know how to treat them.

The simplest such system is a set of two masses connected by springs. Here is a particular realization:



The Lagrangian of this system is

$$L = m\dot{x}_1^2 + \tfrac{1}{2}m\dot{x}_2^2 - \tfrac{1}{2}kx_1^2 - \tfrac{1}{2}k(x_2-x_1)^2$$

The equations of motion are

$$2m\ddot{x}_1 = -kx_1 - k(x_1-x_2)$$
$$m\ddot{x}_2 = -k(x_2-x_1)$$

There is an equilibrium at $(x_1, x_2) = (0, 0)$, and the equations above already give the linearization about this solution. we can organize the equations into the matrix system

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\frac{k}{m}\begin{pmatrix} 1 & -\frac{1}{2} \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

so that A above is given by

$$A = -\frac{k}{m}a \qquad a = \begin{pmatrix} 1 & -\frac{1}{2} \\ -1 & 1 \end{pmatrix}$$

We need to find the right eigenvectors of this matrix. We can first find the eigenvalues ℓ^2 of $\mathbf{a}$, which are the solutions of the *characteristic equation* of $\mathbf{a}$,

$$\det(\underline{a} - \ell^2\underline{1}) = 0$$

Explicitly,

$$\det \begin{pmatrix} 1-\ell^2 & -\frac{1}{2} \\ -1 & 1-\ell^2 \end{pmatrix} = 0$$

Then ℓ^2 obeys

$$(\ell^2-1)^2 - \tfrac{1}{2} = 0$$

for which the solutions are

$$\ell_\pm^2 = 1 \pm \tfrac{1}{\sqrt{2}}$$

Note that

$$\alpha_\pm^2 = -\tfrac{k}{m}\,\ell_\pm^2 < 0$$

so that we have only stable oscillations for this system, as we would expect from our physical intuition. To solve for the corresponding eigenvectors, parametrize them as

$$\eta_\pm = (\text{const}) \cdot \begin{pmatrix} 1 \\ z_\pm \end{pmatrix}$$

The top line of

$$\alpha\,\eta_\pm = \ell_\pm^2\,\eta_\pm$$

is

$$1 - \tfrac{1}{2} z_\pm = \ell_\pm^2 \cdot 1 = 1 \pm \tfrac{1}{\sqrt{2}}$$

and from this we see that

$$z_\pm = \mp \sqrt{2}$$

Thus, we obtain two modes of stables oscillation. The first is

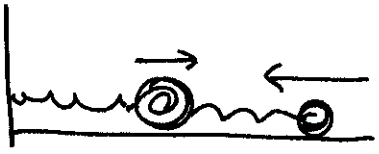
$$\eta_- = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \qquad \omega_- = \left(\frac{k}{m}\right)^{1/2} \left(1 - \frac{1}{\sqrt{2}}\right)^{1/2}$$

This is a motion of the form

The second is

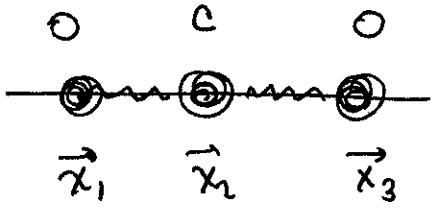
$$\eta_+ = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \qquad \omega_+ = \left(\frac{k}{m}\right)^{1/2} \left(1 + \frac{1}{\sqrt{2}}\right)^{1/2}$$

which gives a motion of the form



with a higher frequency. The general solution of this system is a linear combination of the two oscillations. Note that ζ_1 and ζ_2 are not orthogonal, but they are linearly independent and provide a well-defined basis for expressing the solution that emerges from any possible initial condition.

Here is a slightly harder example: Consider a linear triatomic molecule such as CO_2. The interatomic interactions are not simple springs but arise from a more complicated potential. Nevertheless, there is an equilibrium configuration



We can analyze the small oscillations about this configuration. These oscillations will actually take place in three dimensions, but here I will study only the oscillations in one dimension along the $\hat{x}$ axis along which the molecule is extended. A suitable model Lagrangian is

$$T = \frac{1}{2}m_1(\dot{\vec{x}}_1)^2 + \frac{1}{2}m_2(\dot{\vec{x}}_2)^2 + \frac{1}{2}m_1(\dot{\vec{x}}_3)^2$$

$$- V(|\vec{x}_1 - \vec{x}_2|) - V(|\vec{x}_2 - \vec{x}_3|)$$

I will assume that there is an equilibrium configuration of the three atoms with all three atoms on the $\hat{x}$ axis at the positions

$$x_1^0 = -X \qquad x_2^0 = 0 \qquad x_3^0 = +X$$

Then we can study the small oscillations of the configuration by expanding about this solution

$$x_1 = -X + \xi_1 \qquad x_2 = \xi_2 \qquad x_3 = X + \xi_3$$

It is easiest to perform the linearization directly in the Lagrangian. We substitute the above expressions for the x_i into L and drop all terms beyond those *quadratic* in the ξ_i. The linear terms must vanish, because for any solution of the equations of motion the Lagrangian is unchanged under a linear variation of the coordinates.

The kinetic energy of L is already quadratic in the ξ_i,

$$T = \frac{1}{2} m_1 \dot{\xi}_1^2 + \frac{1}{2} m_2 \dot{\xi}_2^2 + \frac{1}{2} m_1 \dot{\xi}_3^2$$

For the potential energy, we expand

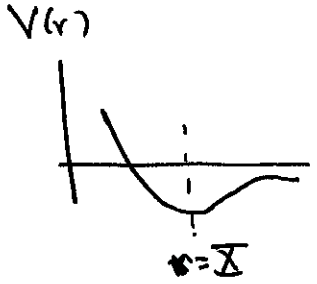
$$V(|x_1 - x_2|) = V(I) + \left.\frac{\partial V}{\partial r}\right|_0 (\xi_2 - \xi_1) + \frac{1}{2} \left.\frac{\partial^2 V}{\partial r^2}\right|_0 (\xi_2 - \xi_1)^2 + \cdots$$

where

$$\left.\frac{\partial V}{\partial r}\right|_0 = \left.\frac{\partial V}{\partial r}\right|_{r=I}$$

At the equilibrium point

$$\left.\frac{\partial V}{\partial r}\right|_{r=I} = 0$$



thus, the terms linear in ξ_i do vanish. Let

$$\left.\frac{\partial^2 V}{\partial r^2}\right|_0 = k$$

then the quadratic terms in V are

$$V = \frac{1}{2} k (\xi_2 - \xi_1)^2 + \frac{1}{2} k (\xi_3 - \xi_2)^2$$

and we find the following Lagrangian,

$$L = \frac{1}{2} m_1 \dot{\xi}_1^2 + \frac{1}{2} m_2 \dot{\xi}_2^2 + \frac{1}{2} m_1 \dot{\xi}_3^2 - \frac{1}{2} k (\xi_2 - \xi_1)^2 - \frac{1}{2} k (\xi_3 - \xi_2)^2$$

which is in fact the Lagrangian of balls along the $\hat{x}$ axis connected by two identical springs.

The equations of motion derived from this Lagrangian are

$$\begin{pmatrix} m_1 \ddot{\xi}_1 \\ m_2 \ddot{\xi}_2 \\ m_1 \ddot{\xi}_3 \end{pmatrix} = -k \begin{pmatrix} \xi_1 - \xi_2 \\ (\xi_2 - \xi_1) + (\xi_2 - \xi_3) \\ \xi_3 - \xi_2 \end{pmatrix}$$

or

$$\begin{pmatrix} \ddot{\xi}_1 \\ \ddot{\xi}_2 \\ \ddot{\xi}_3 \end{pmatrix} = - \begin{pmatrix} k/m_1 & -k/m_1 & 0 \\ -k/m_2 & 2k/m_2 & -k/m_2 \\ 0 & -k/m_1 & k/m_1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$$

It is not difficult to read off the normal modes. Note first that

$$\eta = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

gives a *zero* eigenvalue

$$\lambda_1^2 = 0$$

This mode corresponds to the motion

$$x_i \rightarrow x_i + a$$

which is a uniform translation of the molecule. This motion can proceed freely, with no restoring force.

The symmetry of the molecule motivates us to guess the other two eigenmodes. These are of the form

$$\eta_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

with

$$\lambda_2^2 = -\frac{2k}{m_1}$$

and

$$\eta_3 = \begin{pmatrix} 1 \\ -a \\ 1 \end{pmatrix}$$

with a a constant to be determined. Applying the matrix to this vector, we find the two equations

$$\lambda_3^2 = -k/m_1\,(1+a)$$

$$\lambda_3^2(-a) = -\frac{2k}{m_2}(-a) + \frac{2k}{m_2} = \frac{2k}{m_2}(1+a)$$

Combining these equations, we find

$$a = 2\frac{m_1}{m_2} \qquad \lambda_3^2 = -\frac{k}{m_1}\left(1+2\frac{m_1}{m_2}\right)$$

The general solution for small deviations from equilibrium is then

$$\xi(t) = \eta_1 (A+Bt) + \eta_2\, C\cos(\omega_2 t+\phi_2) + \eta_3\, D\cos(\omega_3 t+\phi_3)$$

where

$$\eta_1 = \begin{pmatrix}1\\1\end{pmatrix} \quad \text{uniform translation}$$

$$\eta_2 = \begin{pmatrix}1\\0\\1\end{pmatrix} \quad \omega_2 = \left(\frac{2k}{m_1}\right)^{1/2}$$

$$\eta_3 = \begin{pmatrix}1\\-2m_1/m_2\\1\end{pmatrix} \quad \omega_3 = \left[\frac{k}{m_1}\left(1+2\frac{m_1}{m_2}\right)\right]^{1/2}$$

There is a similar description of the normal modes of motion perpendicular to the $\hat{x}$ axis. At equilibrium, y_1, y_2, y_3 are zero, so for small deviations, we can write

$$y_1 = \eta_1 \qquad y_2 = \eta_2 \qquad y_3 = \eta_3$$

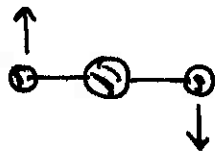
The normal mode

$$\begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

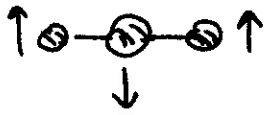
is a translation in the $\hat{y}$ direction and so should have a zero eigenvector. There is another mode that can be identified with a symmetry direction

$$\begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

This is a rotation about the $\hat{z}$ axis



and so it also must have zero restoring force. The third mode is



This mode also has zero restoring force in the oversimplified model we are using here, but in real chemistry it has a finite, nonzero frequency. The situation for $\hat{z}$ oscillations is similar. In all we find 9 modes of small oscillations:

5 with $\omega^2 = 0$ 3 translations + 2 rotations

4 with $\omega^2 > 0$

Notice that a rotation about the $\hat{x}$ axis leaves the equilibrium state *unchanged*, so this operation does not correspond to a mode of small oscillation.

I would like to consider one more example along this line, a long chain of masses connected by potentials. This problem arises in many contexts in physics, for example, in the analysis of motions of a crystal of atoms. The small oscillations from equilibrium can be idealized as a set of masses interacting pairwise through springs

where, for definiteness, I have tethered the springs at the two ends. The equations of motion for this system are

$$
\begin{aligned}
m\ddot{x}_1 &= -kx_1 - k(x_1 - x_2) \\
m\ddot{x}_2 &= -k(x_2 - x_1) - k(x_2 - x_3) \\
&\vdots \\
m\ddot{x}_{N-1} &= -k(x_{N-1} - x_{N-2}) - k(x_{N-1} - x_N) \\
m\ddot{x}_N &= -k(x_N - x_{N-1}) - kx_N
\end{aligned}
$$

The problem of finding normal modes of this system is equivalent to that of finding the eigenvalues and eigenvectors of a large but regular matrix,

$$
\begin{pmatrix} \ddot{x}_1 \\ \vdots \\ \ddot{x}_N \end{pmatrix} = -\frac{k}{m} \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}
$$

How do we solve this problem? First, we can make all of the equations identical by introducing masses $i = 0, N+1$ with $\xi_0 = \xi_{N+1} = 0$. Then the equations are all of the form

$$
\ddot{x}_n = -k(x_n - x_{n-1}) - k(x_n - x_{n+1})
$$

The eigenvalues equations can then be solved by the ansatz

$$\eta_n = e^{ipn}$$

where p is a constant.

$$-\frac{k}{m}(\eta_n - \eta_{n-1}) - \frac{k}{m}(\eta_n - \eta_{n+1})$$

$$= -\frac{k}{m}\left[2e^{ipn} - e^{ip(n-1)} - e^{ip(n+1)}\right]$$

$$= -\frac{k}{m}\left[2 - e^{-ip} - e^{ip}\right] \quad e^{ipn}$$

We recognize the corresponding eigenvalue as

$$\lambda^2 = -\omega^2(p) = -2\frac{k}{m}(1-\cos p)$$

Now we need to find the possible values of p. To do that, we impose the conditions $\xi_0 = \xi_{N+1} = 0$. Since the equations are linear, the imaginary part of the solution above is a solution. Then

$$\eta_n = \sin pn$$

This automatically satisfies $\xi_0 = 0$. To solve the other boundary condition, take

$$p = \frac{\pi m}{N+1} \qquad m = 1, 2, \cdots, N$$

This gives exactly N normal modes, which exhausts the space of eigenvectors.

Let me now turn to a more complex and interesting example, the *Duffing equation*,

$$\ddot{x} = -\dot{x} + x - x^3$$

This is a damped nonlinear oscillator. We can write the equation as a pair of first-order equations

$$\dot{x} = y$$
$$\dot{y} = -y + x - x^3$$

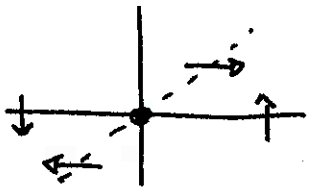
allowing a phase plane analysis similar to those in the first lecture. It will be interesting to combine the phase plane picture with the analysis of stability.

An equilibrium point must have $y = 0$. Then, solving the second equation for $\dot{y} = 0$, we find three solutions

$$x^* = -1, 0, 1$$

We can analyze the behavior in the neighborhood of these points by sketching the phase plane flows, or by a formal stability analysis. Near $x = 0$, the x^3 term is irrelevant, and the phase plane flow is

$$\dot{x} = y$$
$$\dot{y} = -y + x$$



Alternatively, the linearization of the equations gives

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The eigenvectors of the matrix are

$$\eta_+ = \begin{pmatrix} 1 \\ \lambda_+ \end{pmatrix} \qquad \eta_- = \begin{pmatrix} 1 \\ \lambda_- \end{pmatrix}$$

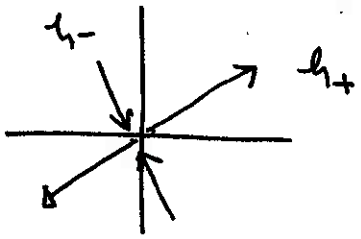
corresponding to the eigenvalues

$$\lambda_\pm = \pm\sqrt{\frac{5}{4}} - \frac{1}{2}$$

The solutions of the linearized equations are

$$\begin{pmatrix} x \\ y \end{pmatrix} = A_+ \eta_+ e^{\lambda_+ t} + A_- \eta_- e^{\lambda_- t}$$

and this gives the flow pattern

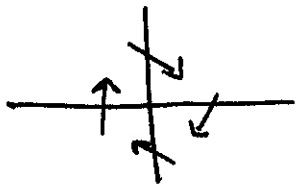


near $x = y = 0$. Note that the flow is unstable in the λ_+ direction.

Near $x = 1$, write $x = 1 + \xi$. The linearized equations are

$$\dot{\xi} = y$$
$$\dot{y} = -y - 2\xi$$

We can sketch the phase plane flow as



Alternatively, we can write the linearized equations as

$$\begin{pmatrix} \dot{\xi} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} \xi \\ y \end{pmatrix}$$

The eigenvectors of the matrix are

$$\eta_+ = \begin{pmatrix} 1 \\ \lambda_+ \end{pmatrix} \qquad \eta_- = \begin{pmatrix} 1 \\ \lambda_- \end{pmatrix}$$

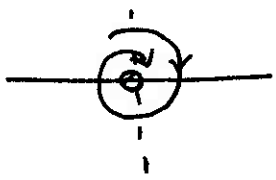
corresponding to the eigenvalues

$$\lambda_\pm = \pm i\left(\frac{7}{4}\right)^{\frac{1}{2}} - \frac{1}{2}$$

The solutions of the linearized equations are

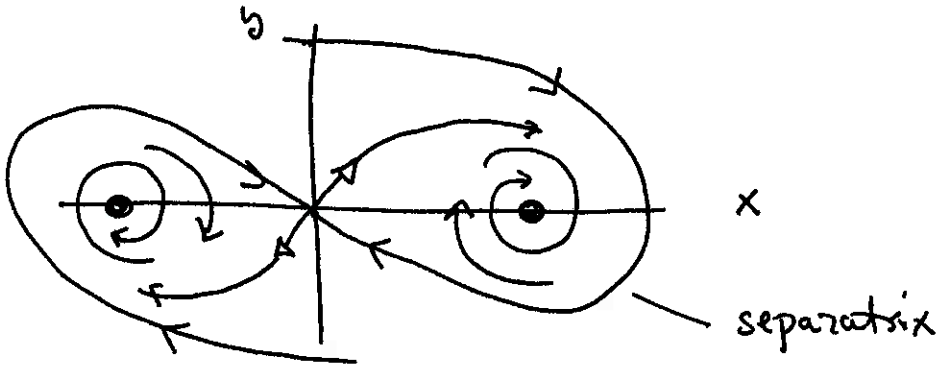
$$\begin{pmatrix} \xi \\ y \end{pmatrix} = \left(A_+ \eta_+ e^{i\frac{\sqrt{7}}{2}t} + A_- \eta_- e^{-i\frac{\sqrt{7}}{2}t}\right) e^{-\frac{1}{2}t}$$

Both solutions are damped with time, so give the flow pattern



near $x = 1$, $y = 0$. A similar analysis holds near $x = -1$, $y = 0$.

It is an interesting exercise to join up these the behaviors near these three points to visualize the global flow pattern in the phase plane. Here is the result:



Notice that there is a curve, called the *separatrix*, that divides the phase plane into two distinct regions. One is attracted to the equilibrium point at $x = 1$; the other is attracted to the fixed point at $x = -1$.