

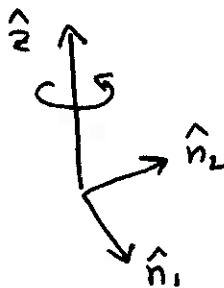
Motion in a Rotating Coordinate System

An important problem related to the analysis of constrained systems is that of particle motion with respect to a rotating coordinate system. This problem appears in many contexts in physics, in the motion of particles or fluids on the surface of the earth, in astronomical systems, and in the motions of molecules and heavy nuclei. In this lecture, I will analyze this problem with the help of Lagrangian mechanics.

Consider a system consisting of one particle moving in 3 dimensions and interacting with a potential V , which might be due to other particles. The Lagrangian is

$$\mathcal{L} = \frac{1}{2} m (\dot{\vec{r}})^2 - V(\vec{r})$$

Now parametrize \vec{r} so that the motion is measured with respect to a coordinate system that is in uniform rotation about the \hat{z} axis. A set of basis vectors for this coordinate system is $(\hat{n}_1, \hat{n}_2, \hat{z})$,



where, with respect to an orthonormal inertial frame,

$$\hat{n}_1 = (\cos \Omega t, \sin \Omega t, 0)$$

$$\hat{n}_2 = (-\sin \Omega t, \cos \Omega t, 0)$$

$$\hat{z} = (0, 0, 1)$$

Note that the dot products of the basis vectors are time-independent,

$$(\hat{n}_1)^2 = 0, \quad \hat{n}_1 \hat{n}_2 = 0 \quad \Rightarrow \quad \hat{n}_1 \cdot \dot{\hat{n}}_1 = 0$$

and that

$$\dot{\hat{n}}_1 = \Omega \hat{n}_2 \quad \dot{\hat{n}}_2 = -\Omega \hat{n}_1$$

We can expand \vec{r} in this basis,

$$\vec{r} = x \hat{n}_1 + y \hat{n}_2 + z \hat{z}$$

and compute

$$\dot{\vec{r}} = (\dot{x} - \Omega y) \hat{n}_1 + (\dot{y} + \Omega x) \hat{n}_2 + \dot{z} \hat{z}$$

Then the Lagrangian above becomes

$$\mathcal{L} = \frac{m}{2} \left[(\dot{x} - \Omega y)^2 + (\dot{y} + \Omega x)^2 + \dot{z}^2 \right] - V(\vec{r})$$

In the following, I will assume that $V(x, y, z)$ has no explicit dependence on t . This will be true if V is only a function of r or, more generally, if it is symmetrical about the \hat{z} axis. V will also be independent of explicit t -dependence if it is produced by other particles whose positions are also measured with respect to the rotating coordinates. For example, in a 2-body system with particles interacting through central forces

$$V = V(|\vec{r}_1(t) - \vec{r}_2(t)|)$$

The individual arguments, for example, \vec{r}_1 , are time-dependent when written in terms of coordinates (x_1, y_1, z_1) , but the distances

$$|\vec{r}_1(t) - \vec{r}_2(t)|$$

are not.

We can now work out the equations of motion. The conjugate momenta to (x, y, z) are

$$P_x = m(\dot{x} - \Omega y) \quad P_y = m(\dot{y} + \Omega x) \quad P_z = m\dot{z}$$

If we write

$$\vec{x} = (x, y, z)$$

for the coordinates in the rotating coordinate system, and set

$$\vec{\Omega} = \Omega \hat{z}$$

then we can write the momenta as

$$\vec{p} = m(\dot{\vec{x}} + \vec{\Omega} \times \vec{x})$$

The Euler-Lagrange equations for the components of \vec{x} then read

$$\frac{d}{dt} (m(\dot{x} - \Omega y)) = m\Omega(\dot{y} + \Omega x) - \frac{\partial V}{\partial x}$$

$$\frac{d}{dt} (m(\dot{y} + \Omega x)) = -m\Omega(\dot{x} - \Omega y) - \frac{\partial V}{\partial y}$$

$$\frac{d}{dt} (m\dot{z}) = -\frac{\partial V}{\partial z}$$

These equations can be rearranged as follows:

$$m\ddot{x} = -\frac{\partial V}{\partial x} + m\Omega^2 x + 2m\Omega\dot{y}$$

$$m\ddot{y} = -\frac{\partial V}{\partial y} + m\Omega^2 y - 2m\Omega\dot{x}$$

$$m\ddot{z} = -\frac{\partial V}{\partial z}$$

or

$$m\ddot{\vec{x}} = -\frac{\partial V}{\partial \vec{x}} + \vec{F}_{\text{cent.}} + \vec{F}_{\omega}$$

Two new force terms appear that are not present in an inertial frame. These are

$$\vec{F}_{\text{cent.}} = m\Omega^2 [\vec{x} - \hat{\Omega}\hat{\Omega}\cdot\vec{x}] = -m\vec{\Omega} \times (\Omega \times \vec{x})$$

and

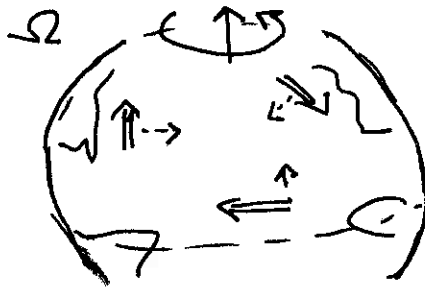
$$\vec{F}_{\omega} = -2m\vec{\Omega} \times \dot{\vec{x}}$$

The first of these is the *centrifugal force*. This points outward in the plane perpendicular to $\vec{\Omega}$ and has magnitude

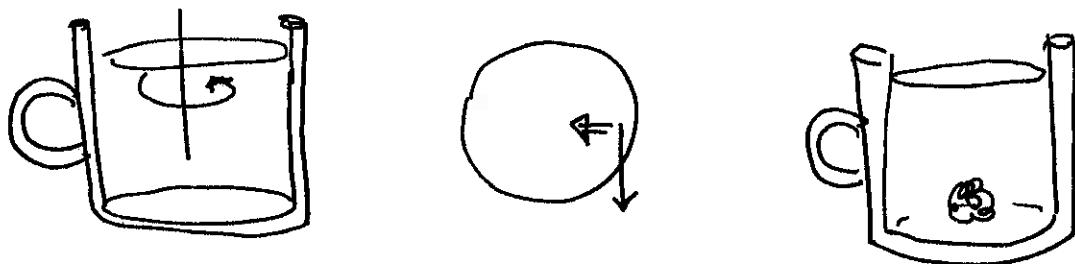
$$m\Omega^2 |x_{\perp}| = \frac{m|v_{\perp}|^2}{x_{\perp}} = \frac{mv^2}{r}$$

When you teach first-year mechanics, you teach your students that this is an illusion, but it has real motive effects with respect to the rotating coordinate system. This centrifugal force is very easy to understand intuitively.

The second new force is the *Coriolis force*. This is a much more mysterious and less visualizable force. However, it has profound implications. The Coriolis force is responsible for the large-scale circulation of fluids in the oceans and atmospheres.



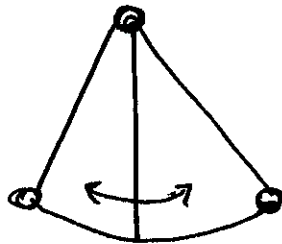
It also is the reason that, when you stir a cup of tea, the tea leaves end up in the center.



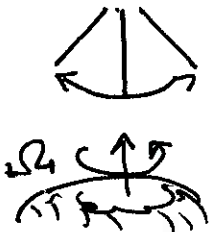
These and other implications of the Coriolis force in fluid dynamics will be discussed in your continuum mechanics course.

In this lecture, I will confine my discussion to effects of the centrifugal and Coriolis forces in particle mechanics problems. I will discuss two illustrative examples.

The first of these is the *Foucault pendulum*, a pendulum whose swinging is influenced by the rotation of the earth. A pendulum at rest in an inertial frame will oscillate continually in a single plane.

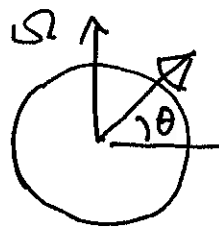
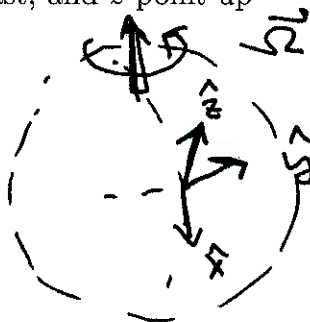


For a pendulum positioned on the rotating earth, however, the plane of the swinging motion will rotate with respect to the earth's coordinate system. This is easy to see for a pendulum positioned at the North Pole. The pendulum oscillates in a plane fixed with respect to an inertial frame, while the earth rotates under it. The plane in which the pendulum swings rotates with respect to the earth by 2π in 24 hours.



What happens at other latitudes is less obvious and requires some analysis.

We can formulate the problem in coordinates at the location of the pendulum. Let \hat{x} point south, \hat{y} point east, and \hat{z} point up



Since the pendulum has fixed length l , the coordinates of its position satisfy

$$[x^2 + y^2 + (z-l)^2] = l^2$$

The potential energy of the pendulum is then

$$V = mgz = mg \left[l - (l^2 - (x^2 + y^2))^{\frac{1}{2}} \right]$$

I will consider only small oscillations; thus, I will approximate this expression by

$$V \approx \frac{1}{2} \frac{mg}{l} (x^2 + y^2)$$

The rotation of the earth is represented as a vector $\vec{\Omega}$. At latitude θ ,

$$\vec{\Omega} = (-\Omega \cos \theta, 0, \Omega \sin \theta)$$

$$\vec{\Omega} \times \vec{x} = \Omega (-\sin \theta y, \sin \theta x, -\cos \theta z) + \mathcal{O}(z)$$

Using this vector, we can write the Lagrangian of the system as

$$L = \frac{1}{2} m (\dot{\vec{x}} + \vec{\Omega} \times \vec{x})^2 - V$$

or, more explicitly,

$$L = \frac{1}{2} m \left\{ (\dot{x} - \Omega \sin \theta y)^2 + (\dot{y} + \Omega \sin \theta x)^2 + (\dot{z} - \Omega \cos \theta y)^2 + \mathcal{O}(x, y, z) \right\} - V$$

Since

$$z = \mathcal{O}(x^2, y^2)$$

terms such as xz or yz are of order x^3 or y^3 and so z can be neglected in a small-oscillation analysis. Thus, the Lagrangian reduces to

$$\mathcal{L} = \frac{1}{2} m \left[(\dot{x} - \Omega \sin \theta y)^2 + (\dot{y} + \Omega \sin \theta x)^2 \right] - V(x, y)$$

This system is equivalent to a pendulum swinging in a coordinate system that rotates about a vertical axis with angular velocity

$$\omega_F = \Omega \sin \theta$$

By our earlier argument involving a pendulum at the North Pole of the earth, this must be the angular velocity of rotation of the plane of the pendulum's motion.

The equations of motion that follow from the last form of the Lagrangian are

$$\begin{aligned} m \ddot{x} &= -\frac{mg}{l} x + 2m \Omega \sin \theta \dot{y} \\ m \ddot{y} &= -\frac{mg}{l} y - 2m \Omega \sin \theta \dot{x} \end{aligned}$$

We see the effect of the Coriolis forces explicitly as causing a rotation of the pendulum's motion with respect to the coordinate system fixed in the earth. It is not difficult to solve these linear equations. Write

$$\underline{X} = x + iy$$

Then the two equations become

$$m \ddot{\underline{X}} = -m \frac{g}{l} \underline{X} - i 2m \Omega \sin \theta \dot{\underline{X}}$$

We can look for complex-valued solutions of the form

$$\underline{x} = A e^{-i\omega t}$$

Inserting this formula, the equation becomes

$$\left(-m\omega^2 = -m\frac{g}{l} - 2m\Omega \sin\theta \omega \right) e^{-i\omega t}$$

So ω must satisfy

$$\omega^2 = 2\Omega \sin\theta \omega + \frac{g}{l}$$

The solutions of this equation are

$$\begin{aligned} \omega_{\pm} &= \Omega \sin\theta \pm \left[\frac{g}{l} + \Omega^2 \sin^2\theta \right]^{\frac{1}{2}} \\ &\approx \Omega \sin\theta \pm \left(\frac{g}{l} \right)^{\frac{1}{2}} \quad \Omega \ll \frac{g}{l} \end{aligned}$$

A typical solution of the motion is

$$\underline{x}(t) = A \left(e^{-i\omega_+ t} + e^{-i\omega_- t} \right)$$

or

$$x(t) = 2A \cos \omega_F t \cos \left(\frac{g}{l} \right)^{\frac{1}{2}} t$$

$$y(t) = -2A \sin \omega_F t \cos \left(\frac{g}{l} \right)^{\frac{1}{2}} t$$

This is an oscillation at the frequency $(g/\ell)^{1/2}$ that the pendulum would have in an inertial frame, superposed on a rotation of the plane of the oscillation at the frequency

$$-\omega_F = -\Omega \sin \Theta$$

Note that the plane of oscillation rotates *backward* with respect to the direction of $\vec{\Omega}$.

The Foucault pendulum gives a beautiful visualization of the rotation of the earth. Foucault set up his pendulum in the Pantheon in Paris as a public demonstration. Today, most good science museums (for example, the Academy of Sciences in Golden Gate Park) will feature this demonstration.

My second example of the forces that arise in a rotating coordinate system is the *restricted 3-body problem* of celestial mechanics. The standard *3-body problem* is the problem of 3 point masses m_1, m_2, m_3 interacting through gravitational forces. This is a famously difficult problem, and most likely it has no general closed-form solution using our current mathematical concepts. We can make the problem easier by assuming that $m_3 \ll m_1, m_2$. Then it is a good approximation to solve for the motion of bodies 1 and 2—which is reducible to a 1-body central force problem—and then solve for the motion of 3 in the resulting time-dependent gravitational potential. For simplicity, I will assume that the relative position vector

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

follows a circular orbit, and that the motion of 3 takes place in the plane of that orbit. The problem with these simplifications is a good first approximation to the earth-moon-sun system, and also to the motion of the *Trojan asteroids* which interact with the sun and Jupiter.

To begin, I will give a parametrization of the motion of 1 and 2. Let $m_1 \geq m_2$, and define

$$m = \frac{m_2}{m_1 + m_2} \quad 1 - m = \frac{m_1}{m_1 + m_2}$$

Work first in an inertial frame in which the center of mass of 1 and 2 is at rest. The relative coordinate $\vec{r} = \vec{r}_1 - \vec{r}_2$ obeys

$$\mu \ddot{\vec{r}} = - \frac{G m_1 m_2}{r^2} \hat{r} \quad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

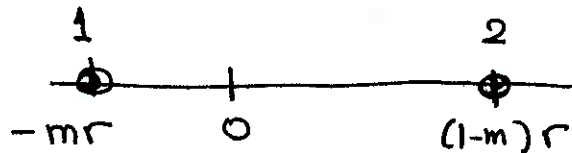
$$\ddot{\vec{r}} = - \frac{G (m_1 + m_2)}{r^2} \hat{r}$$

We found the general solution to this equation earlier in the course. In the solution with circular motion, the two masses revolve about the center of mass with frequency Ω given by

$$\Omega^2 = \frac{G (m_1 + m_2)}{r^3}$$

Take $\vec{\Omega}$ to point in the \hat{z} direction.

Now transform to a coordinate system that rotates with 1 and 2. In this system, we can place 1 and 2 on the \hat{x} axis



with the center of mass at the origin. I will restrict 3 to move in the (\hat{x}, \hat{y}) plane. Then the equations of motion are

$$m_3 \ddot{x} = m_3 \Omega^2 x + 2 m_3 \Omega \dot{y} - \frac{\partial V}{\partial x}$$

$$m_3 \ddot{y} = m_3 \Omega^2 y - 2 m_3 \Omega \dot{x} - \frac{\partial V}{\partial y}$$

where

$$V = - \frac{G m_1 m_3}{r_{13}} - \frac{G m_2 m_3}{r_{23}}$$

with

$$r_{13} = |\vec{r}_1 - \vec{r}_3| \quad r_{23} = |\vec{r}_2 - \vec{r}_3|$$

We can divide through by m_3 . Next, choose units to simplify the algebra. Measure distance in units of r and time in units of Ω^{-1} by letting

$$\underline{X} = \xi + i\eta = \frac{x + iy}{r}$$

be a function of

$$\tau = \Omega t = \left[\frac{G(m_1 + m_2)}{r^3} \right]^{\frac{1}{2}} t$$

In these coordinates, the masses 1 and 2 are located at $x = -m$ and $x = (1 - m)$, so that

$$\frac{r_{13}}{r} = |\underline{X} + m| \quad \frac{r_{23}}{r} = |\underline{X} - (1 - m)|$$

and

$$V = - \Omega^2 r^2 \left[\frac{1 - m}{|\underline{X} + m|} + \frac{m}{|\underline{X} - (1 - m)|} \right]$$

If we denote

$$\frac{d}{d\tau} A = \dot{A}$$

Ω scales out and we find the equations

$$\ddot{\xi} = 2\dot{\eta} = \xi - \frac{\partial}{\partial \xi} U$$

$$\ddot{\eta} + 2\dot{\xi} = \eta - \frac{\partial}{\partial \eta} U$$

with

$$U = - \frac{1-m}{|\underline{x}+m|} - \frac{m}{|\underline{x}-1+m|}$$

Much is known about the solutions to these equations. For the moment, I will simply look for the stationary solutions to these equations. That is, I will look for solutions in which the body 3 is at rest in the frame rotating with 1 and 2, so that the three bodies form a rigid system in uniform rotation. Then we would have

$$\dot{\xi} = \dot{\eta} = 0$$

It can be shown that there are no such equilibrium points in outside the (\hat{x}, \hat{y}) plane, so this analysis will give the complete classification of these points.

Consider first solutions with $\eta \neq 0$. Then the equations above become

$$0 = \xi - \frac{(1-m)(\xi+m)}{r_1^3} - \frac{m(\xi-1+m)}{r_2^3}$$

$$0 = \eta - \frac{(1-m)\eta}{r_1^3} - \frac{m\eta}{r_2^3}$$

where

$$r_1 = |\mathbf{X} + \mathbf{m}| \quad r_2 = |\mathbf{X} - \mathbf{1} + \mathbf{m}|$$

representing the balance of gravitational and centrifugal forces. We can rewrite these equations as

$$0 = (1-m) \frac{\Omega^2 + m}{r_1} \left[r_1 - \frac{1}{r_1^2} \right] + m \frac{\Omega^2 - 1 + m}{r_2} \left[r_2 - \frac{1}{r_2^2} \right]$$

$$0 = (1-m) \frac{\eta}{r_1} \left[r_1 - \frac{1}{r_1^2} \right] + m \frac{\eta}{r_2} \left[r_2 - \frac{1}{r_2^2} \right]$$

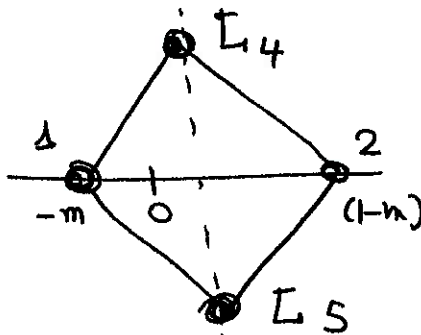
If $\eta \neq 0$, these equations can be satisfied only if

$$\left(r_1 - \frac{1}{r_1^2} \right) = 0 \quad \text{and} \quad \left(r_2 - \frac{1}{r_2^2} \right) = 0$$

so that

$$r_1 = 1 \quad r_2 = 1$$

These conditions give the *Lagrange* equilibrium points

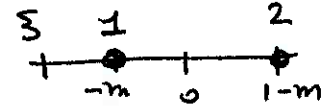


in which the three bodies form an equilateral triangle that rotates rigidly.

There are further equilibrium points on the axis $\eta = 0$. First consider the case in which ξ lies to the left of body 1. Write

$$\rho = \rho_1 = |\xi + m| = -(\xi + m)$$

$$\rho_2 = 1 + \rho = -(\xi - 1 + m)$$



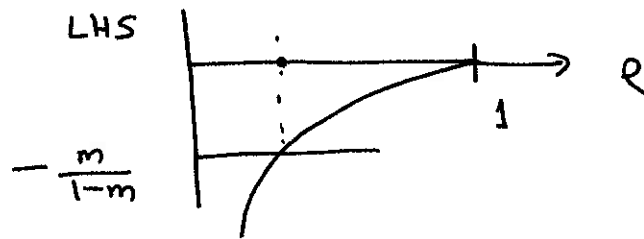
Then the ξ equation becomes

$$0 = (1-m) \left(\rho - \frac{1}{\rho^2} \right) + m \left(1 + \rho - \frac{1}{(1+\rho)^2} \right)$$

or

$$\frac{\left(\rho - \frac{1}{\rho^2} \right)}{(1+\rho) - \frac{1}{(1+\rho)^2}} = - \frac{m}{1-m}$$

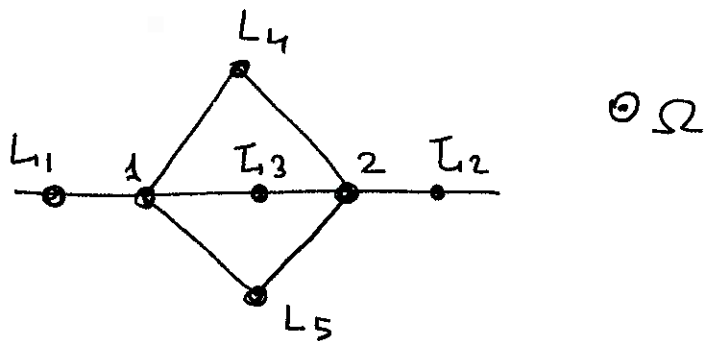
The left-hand side is a monotonically increasing function as ρ is increased from 0 to 1. Thus, there must be exactly one solution between $\rho = 0$ and $\rho = 1$.



Similarly, there is one solution to the ξ equation for $\eta = 0$, $\xi > (1 - m)$, and one more for $\eta = 0$ and

$$-m < \xi < 1-m$$

These five equilibrium solutions are called *libration points*



The solutions L_1 , L_2 , L_3 were discovered by Euler. The less obvious solutions L_4 and L_5 are due to Lagrange. Later in the course, we will study the stability of these equilibrium points. It will turn out that L_1 , L_2 , and L_3 are points of *unstable equilibrium*, as you might expect from your knowledge of electrostatics. However L_4 and L_5 are points of *stable equilibrium*, due to the influence of the Coriolis force.

A remarkable astronomical consequence of this analysis is the existence of the *Trojan asteroids*. These are asteroids present in two clusters on the orbit of Jupiter, located 60° behind and 60° ahead of the line between Jupiter and the sun. These asteroids live stably at the points L_4 and L_5 of the Jupiter-sun gravitational potential. Gerald K. O'Neill, the inventor, with Burt Richter, of the collider-beam technique in particle physics, proposed that space colonies could be set up at the points L_4 and L_5 in the earth-moon system.

The Lagrange point L_2 in the earth-sun system, on a line with the earth and the sun and shielded from the sun by the earth, has become the preferred location for spacecraft measuring the cosmic microwave background. The WMAP satellite has been located in an orbit about L_2 since 2001. Very recently, the Planck satellite observatory has arrived at L_2 and begun its data-taking.