

## Rigid Body Motion

Now that we have analyzed the kinematics of rotational motion, we can apply these results to study rotational dynamics. The idealized rotating object is called a *rigid body*. A rigid body is a collection of particles glued together by constraints so that they form a rigid framework in which  $|\vec{r}_i - \vec{r}_j|$  is fixed for all  $i, j$ . Such a framework has only 6 degrees of freedom, the 3 degrees of freedom of translation of the center of mass, and 3 degrees of freedom for rotations that leave the center of mass fixed.

I will now construct the Lagrangian and the equations of motion of a rigid body. In the following, I will treat the body as a discrete collection of points  $i$  with positions  $r_i$  and masses  $m_i$ . However, the systems we treat as rigid bodies are more typically solid bodies with a continuous distribution of mass characterized by a mass density  $\rho(\vec{r})$ . To obtain formulae for solid bodies, we can pass to a continuum limit

$$\sum_i m_i \quad \rightarrow \quad \int d^3r \rho(\vec{r})$$

The Lagrangian for a rigid body is given by

$$L = T - V + \sum_i \lambda_i (\text{constraint})_i$$

In my analysis, I will consider only the 6 degrees of freedom that satisfy the constraints. We can neglect the explicit constraint terms in writing the equations of motion for these parameters. The potential energy  $V$  potentially depends how exactly forces act on the body, and possibly also on the shape of the body. The kinetic energy  $T$  can be described more generally, and this is a good place to start the analysis.

Consider first a rigid body with one point fixed. I will take this point to be  $\vec{r} = 0$ , but not necessarily the center of mass of the body. Still, with  $\vec{r} = 0$  fixed, the most general motion of the body is an  $SO(3)$  rotation

$$\vec{r}_i = \vec{R}(t) \cdot \vec{r}_i$$

Let  $\vec{r}_i(0)$  be the initial position of each point mass. The position of the mass at time  $t$  is

$$\vec{r}_i(t) = \vec{R}(t) \cdot \vec{r}_i(0)$$

The velocity of the point is

$$\dot{\vec{r}}_i(t) = \dot{\vec{R}} \cdot \vec{r}_i(0)$$

which we know can be represented as

$$\dot{\vec{r}}_i(t) = \vec{\omega} \times \vec{r}_i(t)$$

with  $\vec{\omega}$  the instantaneous angular velocity. Then the kinetic energy is

$$\begin{aligned} T &= \sum_i \frac{1}{2} m_i (\dot{\vec{r}}_i)^2 \\ &= \sum_i \frac{1}{2} m_i (\vec{\omega} \times \vec{r}_i)^2 \\ &= \sum_i \frac{1}{2} m_i \epsilon^{abc} \omega^b r_i^c \epsilon^{ade} \omega^d r_i^e \end{aligned}$$

or

$$T = \sum_i \frac{1}{2} m_i (r_i^2 \delta^{ab} - r_i^a r_i^b) \omega^a \omega^b$$

It is convenient to write this expression as

$$T = \frac{1}{2} I^{ab} \omega^a \omega^b$$

where

$$I^{ab} = \sum_i m_i (r_i^2 \delta^{ab} - r_i^a r_i^b)$$

is the *inertia tensor* of the body. For a solid body,

$$I^{ab} = \int d^3r \rho(\vec{r}) (r^2 \delta^{ab} - r^a r^b)$$

In either case,  $I_{ab}$  is a  $3 \times 3$  real symmetric matrix. If the rigid body is in instantaneous rotation about the  $\hat{z}$  axis, and  $\phi$  gives the orientation so that  $\vec{\omega} = (0, 0, \dot{\phi})$ , then

$$T = \frac{1}{2} I^{zz} (\dot{\phi})^2$$

When we construct  $L$ , this term will give the dependence on  $\dot{\phi}$ . The conjugate momentum to  $\phi$  will then be

$$I^{zz} \dot{\phi}$$

From our discussion of Noether's theorem, this quantity must be the  $\hat{z}$  component of angular momentum  $L^z$ , and  $L^z$  will be conserved if  $V$  is independent of  $\phi$ .

More generally, if  $\alpha$  gives the axis and angle of an infinitesimal rotation

$$\dot{\vec{r}} = \dot{\alpha} \times \vec{r}$$

Then

$$\vec{L} = \frac{\partial T}{\partial \dot{\vec{r}}} = \frac{\partial T}{\partial \dot{\alpha}} \vec{r}$$

so that

$$L^a = I^{ab} \omega^b$$

The Euler-Lagrange equations for a rigid body with one point fixed then take the form

$$\frac{d}{dt} L^a = - \frac{\partial V}{\partial \alpha^a}$$

If  $V$  is written as a function of the  $\vec{r}_i$ ,

$$\begin{aligned} - \frac{\partial V}{\partial \alpha^a} &= \sum_i \frac{\partial}{\partial \alpha^a} (\dot{\alpha} \times \vec{r}_i) \cdot \left( - \frac{\partial V}{\partial \vec{r}_i} \right) \\ &= \frac{\partial}{\partial \alpha^a} \dot{\alpha} \cdot \sum_i \vec{r}_i \times \left( - \frac{\partial V}{\partial \vec{r}_i} \right) \\ &= \sum_i \left( \vec{r}_i \times \frac{\partial V}{\partial \vec{r}_i} \right)^a \end{aligned}$$

In the last line, I have used the expression for the force  $F_i$  on the body  $i$ ,

$$\vec{F}_i = - \frac{\partial V}{\partial \vec{r}_i}$$

The expression

$$\vec{\tau} = \sum_i \vec{r}_i \times \vec{F}_i$$

is called the *torque* about  $\vec{r}' = 0$ . The the Euler-Lagrange equations read

$$\frac{d}{dt} \vec{L} = \vec{\tau} \quad \text{where} \quad \vec{L} = \vec{I} \cdot \vec{\omega}$$

These are a set of highly implicit equations for the motion of a rigid body. In the second half of this lecture, I will work some examples that will clarify how to work with these equations.

Before we consider dynamical examples, however, we need to discuss the inertia tensor  $I_{ab}$  a bit more. First, I will compute  $I_{ab}$  for some relevant examples.  $I_{ab}$  is a real symmetric matrix, we are especially concerned with the values of its eigenvectors. Quite generally,  $I_{ab}$  can be diagonalized by

$$\mathbf{I} = \mathbf{R} \begin{pmatrix} I_1 & & \\ & I_2 & \\ & & I_3 \end{pmatrix} \mathbf{R}^{-1}$$

where  $R$  is an appropriate matrix in  $SO(3)$ . The eigenvalues of  $I_{ab}$ ,  $I_1$ ,  $I_2$ , and  $I_3$ , are called the *moments of inertia*. By the basic theorems of linear algebra, the three eigenvectors are orthogonal vectors that define a preferred coordinate system fixed in the rigid body. The eigenvalues  $I_j$  are real and positive. To see this, note that, for any vector  $\vec{a}$ ,

$$I^{ab} a^a a^b = \sum_i m_i [r_i^2 a^2 - (\vec{r}_i \cdot \vec{a})^2] \geq 0$$

Equality is true only the degenerate case in which all of the  $\vec{r}_i$  lie along a line. In any other case

$$I^{ab} a^a a^b > 0$$

If  $\vec{a}$  is an eigenvector

$$I^{ab} a^a a^b = I_i a^a a^a = I_i |\vec{a}|^2$$

so the corresponding eigenvalue is positive. It is customary to order the eigenvalues so that

$$I_1 \geq I_2 \geq I_3$$

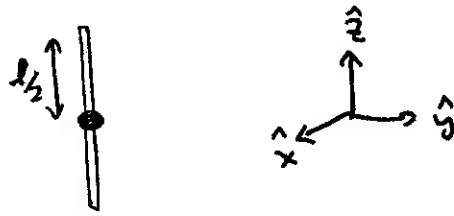
and to choose the axes so that

$$(\hat{1}, \hat{2}, \hat{3})$$

for a right-handed coordinate system.

Here are some simple examples of the computation of the inertia tensor:

First, consider a wire of length  $\ell$ , with  $\vec{r} = 0$  at the center of the wire. Choose the  $\hat{z}$  axis to point along the wire



Then, if  $P = M/\ell$  is the mass density of the wire,

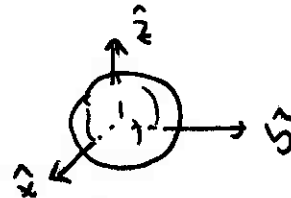
$$I^{ab} = \int_{-l/2}^{l/2} dz P \left[ \underbrace{\begin{pmatrix} z^2 & & \\ & z^2 & \\ & & z^2 \end{pmatrix}}_{r^2 \delta^{ab}} - \underbrace{\begin{pmatrix} 0 & & \\ & 0 & \\ & & z^2 \end{pmatrix}}_{r^a r^b} \right]$$

Then

$$I^{ab} = P \begin{pmatrix} l^3/12 & & \\ & l^3/12 & \\ & & 0 \end{pmatrix} = \frac{M\ell^2}{12} \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}$$

Second, consider a solid sphere of radius  $a$ , with  $\vec{r} = 0$  at the center of the sphere. If  $\rho$  is the mass density in the sphere

$$\rho = \frac{M}{\frac{4}{3}\pi a^3}$$



then

$$I^{ab} = \int_{|\vec{r}| < a} d^3r \rho [r^2 \delta^{ab} - r^a r^b]$$

In a spherically symmetric integral

$$\langle r^a r^b \rangle = \frac{1}{3} \delta^{ab} \langle r^2 \rangle$$

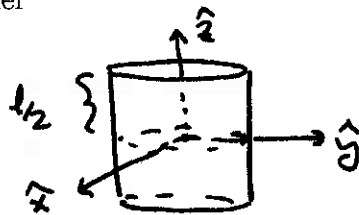
Then

$$\begin{aligned}
 I^{ab} &= \int_{|\vec{r}| < a} d^3r \quad \rho \quad \frac{2}{3} r^2 \delta^{ab} \\
 &= \int_0^a dr \quad r^2 \quad 4\pi \rho \cdot \frac{2}{3} r^2 \delta^{ab} \\
 &= \frac{8\pi}{3} \rho \quad \frac{a^5}{5} \delta^{ab}
 \end{aligned}$$

So the three eigenvalues of  $I_{ab}$  are equal, and all three are equal to

$$I = \frac{2}{5} M a^2$$

Finally, consider a cylinder of radius  $a$  and length  $\ell$ , with  $\vec{r} = 0$  at the center of the cylinder



$$\rho = \frac{M}{\pi a^2 \ell}$$

In the coordinates shown, the inertia tensor is given by the integral

$$\begin{aligned}
 I^{ab} &= \int d^3r \quad \rho \quad \left[ (r^2 + z^2) \left( \frac{\delta^{cd}}{11} \right) \right. \\
 &\quad \left. - \left( \frac{r^c r^d}{r^d z} \mid \frac{r^c z}{z^2} \right) \right] \quad c, d = x, y
 \end{aligned}$$

In a cylindrically symmetric integral

$$\langle r^c r^d \rangle = \frac{1}{2} \delta^{cd} \langle r^2 \rangle \quad \langle r^c z \rangle = 0$$

Then

$$\mathbf{I}^{ab} = \int dr dz \rho \begin{bmatrix} (\frac{1}{2}(r^2+z^2)) \delta^{cd} & 0 \\ 0 & r^2 \end{bmatrix}$$

Since

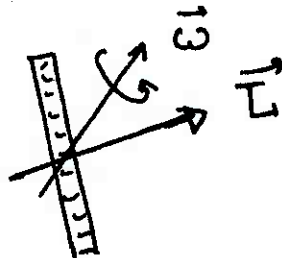
$$\int dr dz r^2 = 2\pi \int_0^a dr \cdot r \cdot r^2 \int_{-l/2}^{l/2} dz = 2\pi \frac{1}{4} a^4 \cdot l$$

$$\int dr dz z^2 = 2\pi \int_0^a dr \cdot r \int_{-l/2}^{l/2} dz z^2 = 2\pi \frac{a^2}{2} \frac{l^3}{12}$$

we find, finally

$$\mathbf{I}^{ab} = M \begin{bmatrix} \frac{1}{4} a^2 + \frac{1}{12} l^2 & & \\ & \frac{1}{4} a^2 + \frac{1}{12} l^2 & \\ & & \frac{1}{2} a^2 \end{bmatrix}$$

In all of these examples, we could find the natural axes in advance using obvious symmetries of the body. In the second case, the eigenvalues of  $I$  are all equal, so all coordinate systems are equivalent, but in the other two cases, the eigenvalues are not equal. Then, if  $\vec{\omega}$  is not parallel to a symmetry direction,  $\vec{\omega}$  and  $\vec{L}$  will not be parallel.



We can easily imagine bodies with no obvious symmetries



However, if such a body is arranged to have all three eigenvalues of the inertia tensor equal, it will behave exactly like a perfect sphere under rotations.

Once we have computed the inertia tensor about one chosen point, the value of the inertia tensor about any other point is obtained from the *parallel-axis theorem*. Typically, it is easiest to compute the inertia tensor about the center of mass of the body, the point

$$\vec{R} = \frac{1}{M} \sum_i m_i \vec{r}_i$$

Let  $I_{CMab}$  be the value of the inertia tensor about the center of mass. If we choose coordinates so that  $\vec{R} = 0$ , this expression is given by

$$I_{CM}^{ab} = \sum_i m_i [r_i^2 \delta^{ab} - r_i^a r_i^b]$$

Then, the value of the inertia tensor about some other point  $\vec{r} = \vec{a}$  is given by

$$\begin{aligned} I^{ab}(\vec{a}) &= \sum_i m_i [(\vec{r}_i - \vec{a})^2 \delta^{ab} - (\vec{r}_i - \vec{a})^a (\vec{r}_i - \vec{a})^b] \\ &= \sum_i m_i [r_i^2 \delta^{ab} - r_i^a r_i^b] + \sum_i m_i (-2\vec{r}_i \cdot \vec{a} \delta^{ab} + r_i^a a^b + a^a r_i^b) \\ &\quad + \sum_i m_i [a^2 \delta^{ab} - a^a a^b] \end{aligned}$$

The second term is zero. Then

$$I^{ab}(\vec{a}) = I_{CM}^{ab} + M [a^2 \delta^{ab} - a^a a^b]$$

The second term in this equation is the value of the inertia tensor about  $\vec{a}$  that we would have found if the entire mass of the body were concentrated at the center of mass.

A similar analysis gives the generalization from a rigid body with one point fixed to a rigid body in both translational and rotational motion. Let  $\vec{R}(t)$  be the position of the center of mass of the body. Let  $\vec{r}_i(t)$  be the position of the point mass  $i$  relative to the center of mass. The allowed motion with respect to the center of mass is only a rotation that leaves the center of mass fixed. Then the velocity of the mass point  $i$  is

$$\dot{\vec{R}}_i = \dot{\vec{R}} + \vec{\omega} \times \vec{r}_i$$



Now we can write the Lagrangian of the system,

$$\begin{aligned} \mathcal{L} &= T - V = \sum_i \frac{1}{2} m_i (\dot{\vec{R}}_i)^2 - V \\ &= \sum_i \frac{1}{2} m_i (\dot{\vec{R}} + \vec{\omega} \times \vec{r}_i)^2 - V \\ &= \sum_i \left( \frac{1}{2} m_i (\dot{\vec{R}})^2 + m_i \dot{\vec{R}} \cdot \vec{\omega} \times \vec{r}_i + \frac{1}{2} m_i (\vec{\omega} \times \vec{r}_i)^2 \right) - V \end{aligned}$$

The second term here is proportional to

$$\sum_i m_i \vec{r}_i = 0$$

Then

$$\mathcal{L} = \frac{1}{2} M (\dot{\vec{R}})^2 + \frac{1}{2} \omega^a I^{ab} \omega^b - V$$

where  $M$  is the total mass of the body and  $I_{ab}$  is the inertia tensor computed about the center of mass. Varying  $L$  with respect to  $\vec{R}(t)$ , we find the equation of motion of the center of mass

$$M\ddot{\vec{R}} = -\frac{\partial}{\partial \vec{R}} V = \sum_i \vec{F}_i$$

Varying  $L$  with respect to a rotation about the center of mass, we find as before

$$\frac{d}{dt} \vec{L} = \vec{\tau} = \sum_i \vec{r}_i \times \vec{F}_i$$

These equations are coupled if  $V$  depends on the orientation of the body, or if the  $\vec{F}_i$  depend on  $\vec{R}$ .

An important special case is motion under gravity. In that case,

$$V = \sum_i m_i g z_i = \sum_i m_i g (\vec{R} + \vec{r}_i) \cdot \hat{z}$$

Since  $\sum_i m_i \vec{r}_i = 0$ , this reduces to

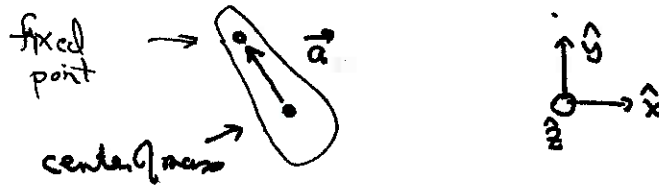
$$V = Mg \vec{R} \cdot \hat{z}$$

Thus, for a rigid body moving under gravity, there is no torque about the center of mass. The equations of motion are

$$M\ddot{\vec{R}} = -Mg \hat{z} \qquad \frac{d}{dt} \vec{L} = 0$$

where  $\vec{L}$  is computed about  $\vec{R}$ .

Now we are ready to discuss the dynamics of rotating bodies. To begin, consider the relatively simple problem of a rigid body constrained to rotate about a fixed axis,



Let  $\vec{a}$  be the vector from the center of mass of the body to the fixed point, and let the body be constrained to rotate only about the  $\hat{z}$  axis through the fixed point. If the rotation angle is  $\phi$ ,

$$\vec{\omega} = \dot{\phi} \hat{z} \quad T = \frac{1}{2} I^{zz} (\dot{\phi})^2$$

Using the parallel-axis theorem

$$I^{zz} = I_{cm}^{zz} + M [|\vec{a}|^2 - (a^z)^2]$$

so

$$I^{zz} = I_{cm}^{zz} + M |\vec{a}_\perp|^2$$

where  $\vec{a}_\perp$  is the vector component of  $\vec{a}$  perpendicular to  $\hat{z}$  and  $I_{CMzz}$  is the relevant component of the inertia tensor computed about the center of mass.

It is convenient to write

$$I^{zz} = M k^2$$

The parameter  $k$  is called the *radius of gyration*. Then

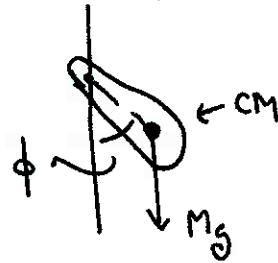
$$T = \frac{1}{2} M k^2 (\dot{\phi})^2$$

Note that, since  $I_{CM\,zz} > 0$ ,

$$k > |\bar{a}_\perp|$$

First, let the body hang as a pendulum, acted on by gravity. The gravitational torque about the attachment point is

$$\tau^z = -|\bar{a}_\perp| M g \sin \phi$$



so the equation of motion is

$$\frac{d}{dt} L^z = \tau^z \quad \Rightarrow \quad M k^2 \ddot{\phi} = -|\bar{a}_\perp| M g \sin \phi$$

This is the equation of a simple pendulum. If  $\phi$  is small,

$$\ddot{\phi} = -\frac{g}{\ell} \phi \quad \text{with} \quad \ell = \frac{k^2}{|\bar{a}_\perp|}$$

The quantity  $\ell$  is the effective length of the pendulum. If the mass of the body is concentrated at the center of mass,  $\ell = |\bar{a}_\perp|$ , but otherwise

$$l > k > |a_{\perp}|$$

The length  $l$  also has another significance. If the body is struck sharply at a point a distance  $l$  from the point of attachment with an impulse  $F\Delta t$ , it receives an angular momentum

$$\Delta L = Fl \Delta t$$

The change in the angular velocity of the body is

$$I \Delta \dot{\phi} = Fl \Delta t$$

which implies

$$\Delta \dot{\phi} = \frac{Fl}{Mk^2} \Delta t$$

The center of mass acquires the additional velocity

$$\Delta v = \frac{Fl}{Mk^2} |a_{\perp}| \Delta t$$

The momentum of the body is

$$P = Mv$$

where  $v$  is the velocity of the center of mass. Thus, the body must receive a momentum

$$\Delta P = \frac{F \ell |a_{\perp}|}{k^2} \Delta t$$

This momentum is not necessarily equal to the  $F\Delta t$  applied. The body is constrained to be fixed at the attachment point. The pivot will apply an additional force at this point to maintain the constraint.



However, if  $\ell = k^2/|\vec{a}_{\perp}|$ , so that the impulse is applied at the center of percussion, the momentum acquired by the body is exactly that supplied by the original impulse, and so contribution from a constraint forces at the pivot is needed. For this reason, the position at the effective length of the pendulum is called the *center of percussion*. A baseball player seeking to hit a ball most effectively will try to hit the ball at the center of percussion of his bat. A pitcher will try to combat this by throwing the ball toward the batter so that he is *jammed* and must strike the ball at a point on the bat too close to his hands.

I will turn now to bodies undergoing free rotation in 3 dimensions. This analysis will apply to rotations of a body in free fall, and also to situations in which a rigid body is rotation about a fixed point such that forces are applied only at that point, so there is no applied torque. In either case, we have

$$\frac{d}{dt} \vec{L} = 0$$

Now we need to turn this equation into a useful set of differential equations for the motion.

The equation  $\vec{L} = 0$  is derived in an inertial frame, relative to axes fixed in space. To solve for the motion, it will be most convenient to rewrite this equation so that it

is referred to the natural set of axes provided by the rigid body, the eigenvectors of the inertia tensor. Let  $\hat{1}$ ,  $\hat{2}$ ,  $\hat{3}$  represent these axes.



These axes are time-dependent, changing with respect to an inertial frame as the body rotates. Since our coordinate system is not an inertial frame, we will find some extra terms in the equations of motion. To work these out, write

$$\vec{L} = L_1 \hat{1} + L_2 \hat{2} + L_3 \hat{3}$$

Then

$$\frac{d}{dt} \vec{L} = \dot{L}_1 \hat{1} + L_1 \dot{\hat{1}} + \dot{L}_2 \hat{2} + L_2 \dot{\hat{2}} + \dot{L}_3 \hat{3} + L_3 \dot{\hat{3}}$$

The time derivatives of the body-fixed axes are given by

$$\dot{\hat{1}} = \vec{\omega} \times \hat{1} \quad \dot{\hat{2}} = \vec{\omega} \times \hat{2} \quad \dot{\hat{3}} = \vec{\omega} \times \hat{3}$$

Then

$$\frac{d}{dt} \vec{L} = \dot{L}_1 \hat{1} + L_1 \dot{\hat{1}} + \dot{L}_2 \hat{2} + L_2 \dot{\hat{2}} + \dot{L}_3 \hat{3} + L_3 \dot{\hat{3}} + \vec{\omega} \times \vec{L}$$

Now use the fact that the axes  $\hat{1}$ ,  $\hat{2}$ ,  $\hat{3}$  are eigenvectors of the inertia tensor. If  $\vec{\omega}$  is specified relative to these axes,

$$L_1 = I_1 \omega_1 \quad L_2 = I_2 \omega_2 \quad L_3 = I_3 \omega_3$$

and

$$\begin{aligned} \vec{\omega} \times \vec{L} &= \hat{1} (\omega_2 L_3 - \omega_3 L_2) + \hat{2} (\omega_3 L_1 - \omega_1 L_3) + \dots \\ &= \hat{1} \omega_2 \omega_3 (I_3 - I_2) + \dots \end{aligned}$$

The components of the equations of motion along the axes  $\hat{1}$ ,  $\hat{2}$ ,  $\hat{3}$  are then

$$\begin{aligned} I_1 \dot{\omega}_1 &= \omega_2 \omega_3 (I_2 - I_3) \\ I_2 \dot{\omega}_2 &= \omega_3 \omega_1 (I_3 - I_1) \\ I_3 \dot{\omega}_3 &= \omega_1 \omega_2 (I_1 - I_2) \end{aligned}$$

These are called *Euler's equations*. They form a set of three coupled first-order equations which can be directly integrated. To understand their content, consider the solution first in the case of a body with cylindrical symmetry

$$I_1 = I_2$$

Then

$$\begin{aligned} I_1 \dot{\omega}_1 &= \omega_2 \omega_3 (I_1 - I_3) \\ I_1 \dot{\omega}_2 &= -\omega_1 \omega_3 (I_1 - I_3) \\ I_3 \dot{\omega}_3 &= 0 \end{aligned}$$

We see that  $\omega_3$  and  $L_3 = I_3 \omega_3$  are constants of the motion. The other two components of  $\vec{\omega}$  obey

$$\dot{\omega}_1 = \frac{\omega_3(I_1 - I_3)}{I_1} \omega_2 \quad \dot{\omega}_2 = -\frac{\omega_3(I_1 - I_3)}{I_1} \omega_1$$

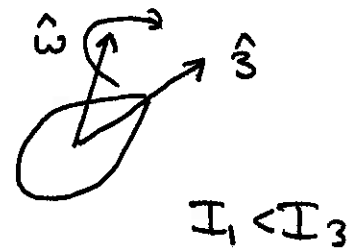
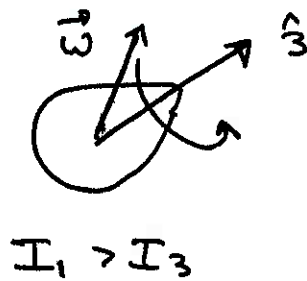
The solutions to these equations are

$$(\omega_1, \omega_2) = \omega_0 \cdot (\sin \Omega t, \cos \Omega t)$$

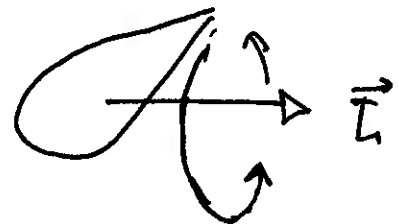
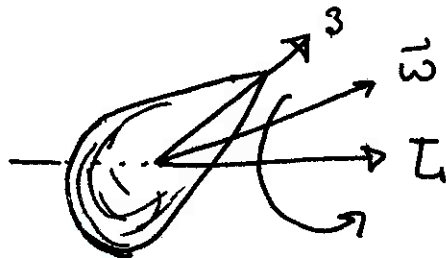
with

$$\Omega = \frac{(I_1 - I_3)}{I_1} \omega_3$$

Thus, in the body-fixed coordinates, the spin precesses about the  $\hat{3}$  axis



With respect to an inertial frame,  $\vec{L}$  maintains a fixed direction. Since  $L_3$  is fixed, the  $\hat{3}$  axis stays at a fixed angle with respect to this direction. The body then *wobbles*, rotating about the fixed axis  $\vec{L}$ , keeping this angle fixed.



Alternatively, we can describe this motion in terms of Euler angles. Choose the initial orientation of the body so that  $\vec{L}$  is parallel to  $\hat{z}$ . Let the Euler angles describe the orientation of the body-fixed axes  $\hat{1}, \hat{2}, \hat{3}$  relative to the axes in the inertial frame, such that, for zero Euler angles these axes are aligned with  $\hat{x}, \hat{y}, \hat{z}$ , respectively. In the previous lecture, we derived

$$\begin{aligned}\vec{\omega} &= \hat{1} [-\dot{\phi} \sin\theta \cos\psi + \dot{\theta} \sin\psi] \\ &+ \hat{2} [\dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi] \\ &+ \hat{3} [\dot{\phi} \cos\theta + \dot{\psi}]\end{aligned}$$

Then, for a body with cylindrical symmetry, the Lagrangian is

$$\begin{aligned}\mathcal{L} &= \frac{1}{2} \omega^a I^{ab} \omega^b \\ &= \frac{1}{2} I_1 [(-\dot{\phi} \sin\theta \cos\psi + \dot{\theta} \sin\psi)^2 + (\dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi)^2] \\ &\quad + \frac{1}{2} I_3 (\dot{\phi} \cos\theta + \dot{\psi})^2 \\ &= \frac{1}{2} I_1 [(\dot{\phi})^2 \sin^2\theta + (\dot{\theta})^2] + \frac{1}{2} I_3 (\dot{\phi} \cos\theta + \dot{\psi})^2\end{aligned}$$

The  $\psi$  equation of motion is very simple

$$\frac{d}{dt} P_\psi = \frac{d}{dt} (I_3 (\dot{\psi} + \dot{\phi} \cos\theta)) = 0$$

This is the equation

$$L_3 = \underline{\text{const}}$$

The  $\phi$  equation is

$$\begin{aligned}
0 &= \frac{d}{dt} p_\phi = \frac{d}{dt} [\mathcal{I}_1 \sin^2 \theta \dot{\phi} + \mathcal{I}_3 \omega_s \theta (\dot{\psi} + \dot{\phi} \cos \theta)] \\
&= \frac{d}{dt} [\sin \theta (\mathcal{I}_1 \dot{\phi} \sin \theta) + \omega_s \theta (\mathcal{I}_3 (\dot{\psi} + \dot{\phi} \cos \theta))]
\end{aligned}$$

This is the equation

$$L_2 = \text{const.}$$

We have assumed that, at time  $t = 0$ ,  $\vec{L}$  points along  $\hat{z}$ . The  $\hat{3}$  axis is at an angle  $\theta$  with respect to the  $\hat{z}$  axis, so

$$L_3 = L \cos \theta$$

Then, from the equations for  $L_3$  and  $L$

$$\mathcal{I}_3 (\dot{\psi} + \dot{\phi} \cos \theta) = L \cos \theta$$

$$\sin^2 \theta \mathcal{I}_1 \dot{\phi} + \cos \theta \mathcal{I}_3 (\dot{\psi} + \dot{\phi} \cos \theta) = L$$

we can solve algebraically for  $\dot{\phi}$  and  $\dot{\psi}$

$$\dot{\phi} = \frac{L}{\mathcal{I}_1} \quad \dot{\psi} = \frac{L}{\mathcal{I}_1 \mathcal{I}_3} (\mathcal{I}_1 - \mathcal{I}_3) \cos \theta$$

Note that the second equation here is

$$\dot{\psi} = \omega_3 \frac{(I_1 - I_3)}{I_1} = \Omega$$

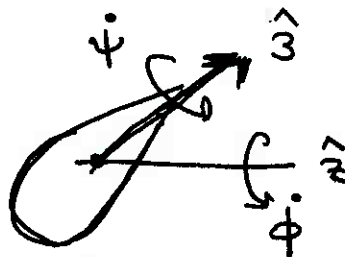
where  $\Omega$  is the precession frequency in the previous analysis. Finally, the  $\theta$  equation is

$$I_1 \ddot{\theta} = I_1 (\dot{\phi})^2 \sin\theta \cos\theta - I_3 \dot{\phi} \sin\theta (\dot{\psi} + \dot{\phi} \cos\theta)$$

If we plug in the formulae for  $\dot{\phi}$  and  $\dot{\psi}$  above and the additional initial condition  $\dot{\theta} = 0$ , we find

$$\ddot{\theta} = 0 \quad \Rightarrow \quad \dot{\theta} = \text{const.}$$

The final picture is



In the case in which all three of the  $I_i$  are distinct, the rotational motion is more complicated. But there is one simple result that is not so difficult to derive. Consider the situation of free rotation about the axis  $\hat{1}$ ,

$$\omega_1 = \Omega \quad \omega_2, \omega_3 = 0$$

constant

This is a solution to the Euler equations. Consider small perturbations about this solution

$$\omega_1 = \Omega + \eta_1(t) \quad \omega_2 = \eta_2(t) \quad \omega_3 = \eta_3(t)$$

where the  $\eta_i(t)$  are to be treated to first order. The Euler equations become

$$I_1 \dot{\eta}_1 = \eta_2 \eta_3 (I_2 - I_3) = \mathcal{O}(\eta^2)$$

$$I_2 \dot{\eta}_2 = \Omega \eta_3 (I_3 - I_1) + \mathcal{O}(\eta^2)$$

$$I_3 \dot{\eta}_3 = \Omega \eta_2 (I_1 - I_2) + \mathcal{O}(\eta^2)$$

Substitute the third equation into the second. This gives

$$I_2 I_3 \ddot{\eta}_2 = \Omega^2 (I_3 - I_1)(I_1 - I_2) \eta_2$$

Then

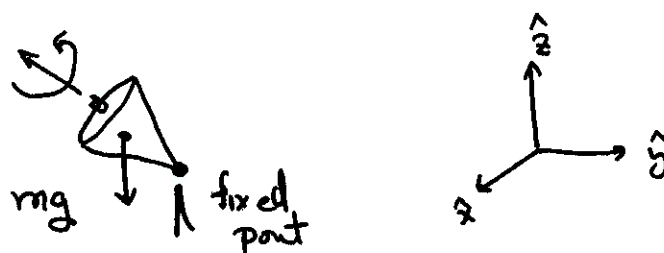
$$\ddot{\eta}_2 = - \left[ \Omega^2 \frac{(I_1 - I_2)(I_1 - I_3)}{I_2 I_3} \right] \eta_2$$

If the quantity in brackets is *positive*, the body will *oscillate* about the state of constant rotation. If the quantity is *negative*, the disturbance will grow exponentially. Thus, the condition of uniform motion about the axis  $\hat{1}$  is *unstable* in the two cases

$$I_1 < I_3, I_1 > I_2 \quad \text{or} \quad I_1 < I_2, I_1 > I_3$$

That is, a body will rotate stably about the axis with the *largest* or the *smallest* moment of inertia, while the rotation about the axis with the intermediate moment of inertia is unstable.

To conclude this discussion, I will consider a system that combines the complexities of anisotropy and motion under an external torque. Consider a symmetrical top, a body with cylindrical symmetry, with one point fixed that is not the center of mass. Gravity will then exert a force on this body that provides a nonzero torque about the fixed point. What is the motion?



The top is described by an inertia tensor about the fixed point

$$\begin{pmatrix} \mathcal{I}_1 & & \\ & \mathcal{I}_1 & \\ & & \mathcal{I}_3 \end{pmatrix}$$

in a body-fixed coordinate system. I will parametrize the orientation of this body in terms of Euler angles. We found earlier, that, for a body with cylindrical symmetry,

$$T = \frac{1}{2} \mathcal{I}^{ab} \omega^a \omega^b = \frac{1}{2} \mathcal{I}_1 [(\dot{\phi})^2 \sin^2 \theta + (\dot{\theta})^2] + \frac{1}{2} \mathcal{I}_3 [\dot{\phi} \cos \theta + \dot{\psi}]^2$$

Also,

$$\hat{z}_3 \cdot \hat{z} = \cos \theta$$

which implies that the gravitational potential energy of the top is

$$V = -Mgl \cos \Theta \quad \text{[Diagram of a top with distance } \ell \text{ from pivot to center of mass]}$$

where  $\ell$  is the distance from the fixed point to the center of mass of the top. The Lagrangian of the top is then

$$\mathcal{L} = \frac{1}{2} I_1 [(\dot{\phi})^2 \sin^2 \Theta + (\dot{\Theta})^2] + \frac{1}{2} I_3 [\dot{\phi} \cos \Theta + \dot{\psi}]^2 - Mgl \cos \Theta$$

This Lagrangian is independent of  $\phi$  and  $\psi$ , and so the corresponding conjugate momenta are conserved,

$$0 = \frac{d}{dt} \mathcal{P}_\psi = \frac{d}{dt} I_3 (\dot{\psi} + \dot{\phi} \cos \Theta)$$

$$0 = \frac{d}{dt} \mathcal{P}_\phi = \frac{d}{dt} [I_1 \sin^2 \Theta \dot{\phi} + I_3 \cos \Theta \dot{\psi} + \dot{\phi} \cos \Theta]$$

The conserved quantities are the components of angular momentum

$$L_3 = I_3 (\dot{\psi} + \dot{\phi} \cos \Theta)$$

$$L_2 = I_1 \sin^2 \Theta \dot{\phi} + L_3 \cos \Theta$$

and we can eliminate  $\dot{\phi}$ ,  $\dot{\psi}$  in terms of these:

$$\dot{\phi} = \frac{L_2 - L_3 \cos \Theta}{I_1 \sin^2 \Theta}$$

$$\dot{\psi} = L_3 \frac{I_1 \sin^2 \Theta + I_3 \cos^2 \Theta}{I_1 I_3 \sin^2 \Theta} - L_2 \frac{\cos \Theta}{I_1 \sin^2 \Theta}$$

Now turn to the  $\theta$  equation. This is

$$\mathcal{I}_1 \ddot{\theta} = \mathcal{I}_1 (\dot{\phi})^2 \sin \theta \cos \theta - \mathcal{I}_3 \dot{\phi} \sin \theta (\dot{\psi} + \dot{\phi} \cos \theta) + Mgl \sin \theta$$

We can substitute for  $\dot{\phi}$  and  $(\dot{\psi} + \cos \theta \dot{\phi})$  to give

$$\mathcal{I}_1 \ddot{\theta} = \frac{\cos \theta}{\mathcal{I}_1 \sin^3 \theta} (\mathcal{L}_2 - \mathcal{L}_3 \cos \theta)^2 - \frac{\mathcal{L}_3}{\mathcal{I}_1 \sin \theta} (\mathcal{L}_2 - \mathcal{L}_3 \cos \theta) + Mgl \sin \theta$$

or

$$\mathcal{I}_1 \ddot{\theta} = \frac{(\mathcal{L}_2 - \mathcal{L}_3 \cos \theta)}{\mathcal{I}_1 \sin^3 \theta} [\cos \theta (\mathcal{L}_2 - \mathcal{L}_3 \cos \theta) - \mathcal{L}_3 \sin^2 \theta] + Mgl \sin \theta$$

or, finally

$$\ddot{\theta} = \frac{\cos \theta (\mathcal{L}_2 - \mathcal{L}_3 \cos \theta)}{\mathcal{I}_1^2 \sin^3 \theta} - \frac{\sin^2 \theta \mathcal{L}_3 (\mathcal{L}_2 - \mathcal{L}_3 \cos \theta)}{\mathcal{I}_1^2 \sin^3 \theta} + \frac{Mgl}{\mathcal{I}_1} \sin \theta$$

This equation can be written as

$$\ddot{\theta} = - \frac{\partial}{\partial \theta} U$$

with

$$U = \frac{(L_2 - L_3 \cos \Theta)^2}{2I_1^2 \sin^2 \Theta} + \frac{Mgl}{I_1} \cos \Theta$$

This form of the equation makes it obvious that the quantity

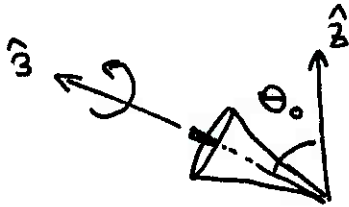
$$\mathcal{E} = \frac{1}{2} \dot{\Theta}^2 + U(\Theta)$$

is a constant of the motion. The total energy of the system is

$$E = I_1 \mathcal{E} + \frac{L_3^2}{2I_3}$$

so the constancy of  $\mathcal{E}$  does follow from energy conservation.

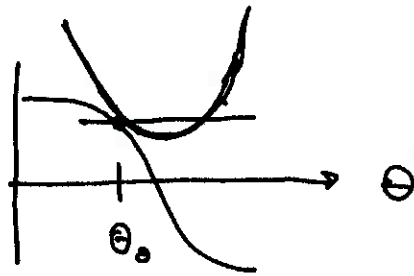
Consider the case in which we give the top an initial orientation of  $\theta_0$  relative to the  $\hat{z}$  axis and an initial spin about its  $\hat{3}$  axis,



Then  $L_z = L_3 \cos \theta_0$  and both quantities are constant. Note, in particular, that  $L_z < L_3$ . The initial value of  $U$  is

$$-U = \frac{L_3^2 (\cos \Theta - \cos \Theta_0)^2}{2I_1^2 \sin^2 \Theta} + \frac{Mgl}{I_1} \cos \Theta$$

This function has the form

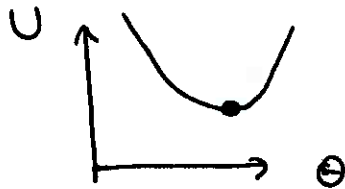


with  $U$  lying above the curve

$$\frac{Mgl}{I_1} \cos \theta$$

and being tangent to this curve at  $\theta = \theta_0$ .

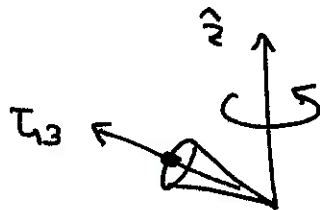
The simplest motion with the given values of  $L_z$  and  $L_3$  is that in which the  $\theta$  position of the top is at the minimum of  $U(\theta)$



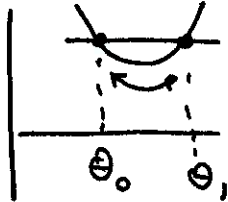
In that case,  $\theta$  would be constant and  $\dot{\phi}$  and  $\dot{\psi}$  would be constant. Since

$$\dot{\phi} = \frac{L_2 - L_3 \cos \theta}{I_1 \sin^2 \theta} = \frac{L_3}{I_1 \sin^2 \theta} (\cos \theta_0 - \cos \theta)$$

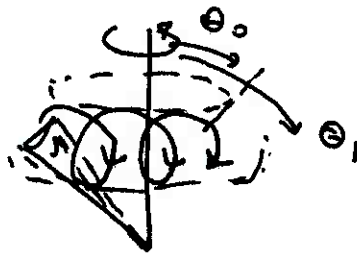
$\dot{\phi}$  will be positive. Then the motion of the top is a smooth *precession* about the  $\hat{z}$  axis



For the initial condition described above, the motion is slightly more complex. The value of  $\theta$  oscillates about the minimum of the potential well provided by  $U$



The motion of the top is then the precession described above and, superposed on it, an oscillation of the  $\theta$  orientation, called *nutation*,



A particular case of interest is that in which the top is initially spinning *vertically*, so that

$$L_2 = L_3 \quad \cos \theta = 1$$

In this case, the effective potential for the motion in  $\theta$  is

$$U = \frac{L_3^2}{2I_1^2} \left( \frac{1 - \cos \theta}{\sin \theta} \right)^2 + \frac{Mgl}{I_1} \cos \theta$$

In this case  $U' = 0$  at  $\theta = 0$ , so there is at least an equilibrium point in this vertical orientation



The second derivative of  $U(\theta)$  at  $\theta = 0$  is

$$U''(\theta=0) = \frac{L_3^2}{4I_1^2} - \frac{Mgl}{I_1}$$

As long as the top spins fast enough, this quantity is positive, and so the equilibrium is stable. But, eventually, the top will be slowed down by friction. When the condition

$$\frac{L_3^2}{4I_1^2} = \frac{Mgl}{I_1}$$

or

$$L_3 = 2 (MglI_1)^{\frac{1}{2}}$$

is met, we have a situation of neutral stability. If the top slows further,  $U(\theta)$  turns over and has a *maximum* at  $\theta = 0$ , and the top falls over.