Nov. 9

Perturbation Theory in Hamiltonian Mechanics

As a final topic in the classical theory of Hamiltonian mechanics, I will recast our previous results on linear stability of static solutions to the equations of mechanics into a Hamiltonian framework. It is interesting to see how, for conservative mechanical systems, our previous results fit together with our general results on structure of Hamiltonian flows.

Let $x_0 = (p_0, q_0)$ be a point in phase space at which

$$\frac{\partial H}{\partial q_i} = 0 \qquad \frac{\partial H}{\partial p_i} = 0$$

so that

$$\dot{q}_i = \dot{p}_i = 0$$

at that point. Then x_0 is an equilibrium point, a static solution to the Hamiltonian equations. We can now consider the evolution of small deviations from this equilibrium point.

Write

$$q_i = q_{0i} + \mathbf{q}_i \qquad p_i = p_{0i} + \mathbf{p}_i$$

and work out the linearized equations of motion for $\mathbf{q}_i$ and $\mathbf{p}_i$. These are

$$\dot{\mathbf{q}}_i = \dot{q}_i = \frac{\partial H}{\partial p_i}\bigg|_0 + \frac{\partial^2 H}{\partial p_i \partial q_j}\bigg|_0 \mathbf{q}_j + \frac{\partial^2 H}{\partial p_i \partial p_j}\bigg|_0 \mathbf{p}_j + \cdots$$

$$\dot{\mathbf{p}}_i = \dot{p}_i = -\frac{\partial H}{\partial q_i}\bigg|_0 - \frac{\partial^2 H}{\partial q_i \partial q_j}\bigg|_0 \mathbf{q}_j - \frac{\partial^2 H}{\partial q_i \partial p_j}\bigg|_0 \mathbf{p}_j + \cdots$$

These equations can be put into matrix form

$$\dot{\begin{pmatrix} q_i \\ p_i \end{pmatrix}} = \begin{pmatrix} \frac{\partial^2 H}{\partial p_i \partial q_j} & \frac{\partial^2 H}{\partial p_i \partial p_j} \\ -\frac{\partial^2 H}{\partial q_i \partial q_j} & -\frac{\partial^2 H}{\partial q_i \partial p_j} \end{pmatrix} \begin{pmatrix} q_j \\ p_j \end{pmatrix}$$

This matrix equation is of the form

$$\dot{\begin{pmatrix} q_i \\ p_i \end{pmatrix}} = \mathcal{K} \begin{pmatrix} q_i \\ p_i \end{pmatrix}$$

and we have seen the matrix $\mathcal{K}$ before as the infinitesimal generator of a general Hamiltonian flow. Thus, it satisfies

$$\mathcal{K}E = (\mathcal{K}E)^T$$

and is a generator of a symplectic matrix. By virtue of this condition, we know that the eigenvalues of $\mathcal{K}$ come either in pairs or in quartets:

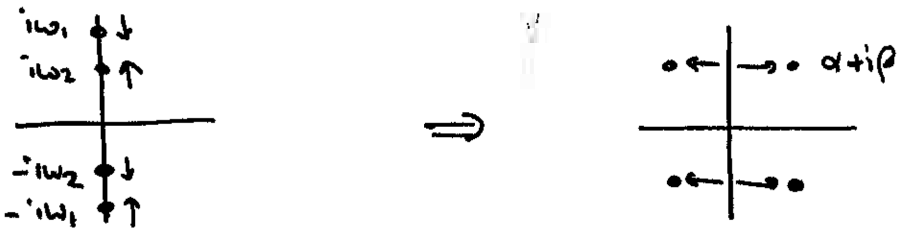
$$\lambda, -\lambda \qquad i\omega, -i\omega \qquad \pm\alpha \pm i\beta$$

Only the second of these choices gives a stable oscillation. Thus, a Hamiltonian system at an equilibrium point can exhibit stable oscillations only if all eigenvalues of $\mathcal{K}$ are pure imaginary.

On the other hand, if we find at a certain level of approximation that the eigenvalues of $\mathcal{K}$ are distinct and pure imaginary, then the oscillations are exactly stable. An eigenvalue can acquire a nonzero real part only as a pair of eigenvalues

$$\pm \alpha \pm i\beta$$

These can evolve from pure imaginary eigenvalues only if two eigenvalues collide:



This means that a Hamiltonian system that appears to be in stable oscillation to the first approximation actually is in exactly stable oscillation except in rather special circumstances. Clearly, we are making strong use here of the fact that Hamiltonian systems naturally have a conserved energy.

A first illustration of this theory is given by the equilibrium point $x = p = 0$ of the harmonic oscillator. For the stable oscillator

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

we have

$$K = \begin{pmatrix} 0 & \frac{1}{m} \\ -m\omega^2 & 0 \end{pmatrix}$$

which has eigenvalues

$$\pm i\omega$$

For the unstable oscillator

$$\mathcal{H} = \frac{p^2}{2m} - \frac{1}{2} m \lambda^2 x^2$$

we find

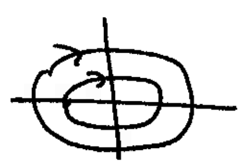
$$K = \begin{pmatrix} 0 & \frac{1}{m} \\ m\lambda^2 & 0 \end{pmatrix}$$

which has eigenvalues

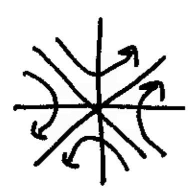
$$\pm \lambda$$

We have discussed these forms of $\mathcal{K}$ as part of our general analysis of symplectic matrices; they are associated with flows of the forms

stable:



unstable:



respectively.

A more interesting example is one that I promised earlier in the course. I will now use Hamiltonian theory to analyze the stability of the libration points in the restricted 3-body problem of celestial mechanics. This is the problem of the motion of a small third body in the gravitational field of two heavier bodies revolving about one another in a circular orbit. This is a nontrivial example, since it involves velocity-dependent forces.

Taking the origin to be the center of mass of the system and the angular velocity to be Ω, and working in the frame rotating with the two heavy bodies, we found for the position (x, y) of the small body the dynamical equations

$$\ddot{x} = \Omega^2 x + 2\Omega \dot{y} - \frac{\partial}{\partial x}(V/m_3)$$
$$\ddot{y} = \Omega^2 y - 2\Omega \dot{x} - \frac{\partial}{\partial y}(V/m_3)$$

where m_3 is the mass of the light body and

$$V/m_3 = -\frac{Gm_1}{r_{13}} - \frac{Gm_2}{r_{23}}$$

Using Kepler's law

$$\Omega^2 = \frac{G(m_1+m_2)}{r_{12}^3}$$

and rescaling time and space appropriately, we converted these equations to simpler ones

$$\ddot{\xi} - 2\dot{\eta} = \xi - \frac{\partial}{\partial \xi}U$$
$$\ddot{\eta} + 2\dot{\xi} = \eta - \frac{\partial}{\partial \eta}U$$

with

$$U = -\frac{(1-m)}{|X+m|} - \frac{m}{|X-(1-m)|}$$

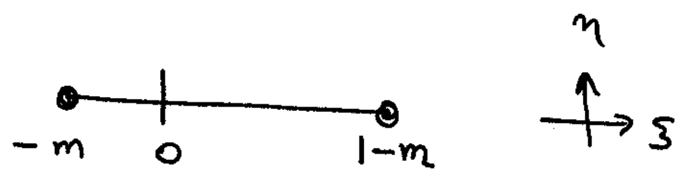
and

$$X = \xi + i\eta$$

The parameter m is given by

$$m = \frac{m_2}{m_1 + m_2} \qquad\qquad m_1 \geq m_2$$

and the two heavy masses are located at



These equations follow from a Hamiltonian. The Hamiltonian is

$$H = \frac{1}{2}\left(p_\xi^2 + p_\eta^2\right) - \left(\xi p_\eta - \eta p_\xi\right) + U(\xi, \eta)$$

The Hamiltonian form of the equations is

$$\dot{\xi} = p_\xi + \eta \qquad\qquad \dot{p}_\xi = p_\eta - \frac{\partial}{\partial \xi} U$$

$$\dot{\eta} = p_\eta - \xi \qquad\qquad \dot{p}_\eta = -p_\xi - \frac{\partial}{\partial \eta} U$$

Then, for example,

$$\ddot{\xi} = \dot{p}_\xi + \dot{\eta} = p_\eta - \frac{\partial U}{\partial \xi} + p_\eta - \xi$$

$$= 2\dot{\eta} + \xi - \frac{\partial U}{\partial \xi}$$

The Hamiltonian can be recast as

$$\mathcal{H} = \frac{1}{2}(p_\xi + \eta)^2 + \frac{1}{2}(p_\eta - \xi)^2 - \Phi(\xi, \eta)$$

where

$$\Phi(\xi,\eta) = \frac{1}{2}(\xi^2 + \eta^2) - U(\xi,\eta)$$

At equilibrium,

$$\dot{\xi} = \dot{\eta} = 0$$

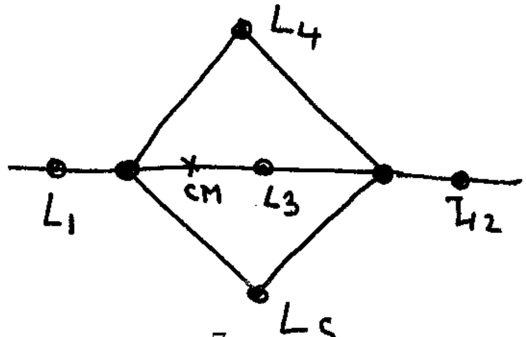
so that

$$p_\xi + \eta = p_\eta - \xi = 0$$

and so

$$\frac{\partial}{\partial \xi}\Phi = \frac{\partial}{\partial \eta}\Phi = 0$$

In our earlier discussion, we solved these equations and found the five Lagrange equilibrium points,



For each of these points, we can compute the stability matrix $\mathcal{K}$. The ingredients are

$$\frac{\partial^2 H}{\partial p_i \partial p_j} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \frac{\partial^2 H}{\partial p_i \partial q_j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\frac{\partial^2 H}{\partial q_i \partial p_j} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \frac{\partial^2 H}{\partial q_i \partial q_j} = \begin{pmatrix} 1-A & -B \\ -B & 1-C \end{pmatrix}$$

where

$$A = \frac{\partial^2 \Phi}{\partial \xi^2} \qquad B = \frac{\partial^2 \Phi}{\partial \xi \partial \eta} \qquad C = \frac{\partial^2 \Phi}{\partial \eta^2}$$

Then the stability matrix takes the form

$$K = \left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ \hline A-1 & B & 0 & 1 \\ B & C-1 & -1 & 0 \end{array} \right)$$

The eigenvalues of $\mathcal{K}$ are the solutions to the characteristic equation

$$\det \left[\begin{array}{cc|cc} -\lambda & 1 & 1 & 0 \\ -1 & -\lambda & 0 & 1 \\ \hline A-1 & B & -\lambda & 1 \\ B & C-1 & -1 & -\lambda \end{array} \right] = 0$$

Computing the determinant, this is the equation

$$\lambda^4 + (4-A-C)\lambda^2 + AC - B^2 = 0$$

This is an equation for λ^2, and so the solutions give pairs of eigenvalues λ, $-\lambda$. The equilibrium points have stable oscillations about them if both solutions for λ^2 are real and *negative*.

To analyze this question, we need to compute A, B, and C. The story is different for the three Euler points L_1, L_2, L_3 from that for L_4, L_5. Consider the Euler points first. We differentiate

$$\Phi = \frac{1}{2}(\xi^2+\eta^2) + \frac{1-m}{r_{13}} + \frac{m}{r_{23}}$$

with

$$r_{13} = |X + m| \qquad r_{23} = |X - (1-m)|$$

and set $\eta = 0$ because the Euler points are on the real axis. The results are written simply in terms of the quantity

$$s = \frac{1-m}{r_{13}^3} + \frac{m}{r_{23}^3}$$

Specifically,

$$A = 1+2s \qquad B = 0 \qquad C = 1-s$$

Then the equations for λ^2 take the form

$$(\lambda^2)^2 + (2-s)(\lambda^2) + (1+2s)(1-s) = 0$$

The product of the eigenvalues is

$$(1+2s)(1-s)$$

so there is a zero eigenvalue at $s = 1$. For $s > 1$, one of the two eigenvalues would be positive and the Euler point would be unstable. It turns out that, at all three of the Euler points, $s > 1$. Then there is always an unstable mode of oscillation.

Here is the analysis of s for the point L_1 to the left of the mass 1. The equation for the location of the point is

$$(1-m)\left(r_{13} - \frac{1}{r_{13}^2}\right) + m\left(r_{23} - \frac{1}{r_{23}^2}\right) = 0$$

with

$$r_{23} = r_{13} + 1$$

Then

$$(1-m)\, r_{13}\left(1 - \frac{1}{r_{13}^3}\right) + m(r_{13}+1) - m\,\frac{(r_{13}+1)}{r_{23}^3} = 0$$

$$r_{13} + m - \frac{m}{r_{23}^3} = r_{13}\left(\frac{1-m}{r_{13}^3} + \frac{m}{r_{23}^3}\right)$$

so that

$$r_{13} + m - \frac{m}{r_{23}^3} = r_{13} \cdot s$$

From the geometry, we see that $r_{23} > 1$. Then

$$s = 1 + \frac{m}{r_{13}}\left(1 - \frac{1}{r_{23}^3}\right) > 1$$

In an similar way, it is possible to show that $s > 1$ for the other two Euler points.

At the Lagrange points L_4 and L_5, $r_{13} = r_{23} = 1$. We can use this to obtain explicit results for A, B, and C. After some algebra, one finds

$$A = \frac{3}{4} \qquad B = -\frac{3\sqrt{3}}{4}(1-2m) \qquad C = \frac{9}{4}$$

The eigenvalue equation is then

$$\lambda^4 + \lambda^2 + \frac{27}{4}m(1-m) = 0$$

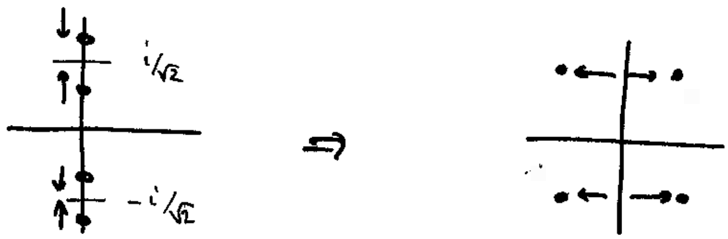
The solutions of this equation are

$$\lambda^2 = -\frac{1}{2} \pm \frac{1}{2}\left[1 - 27m(1-m)\right]^{\frac{1}{2}}$$

This gives stable oscillations as long as

$$27m(1-m) < 1$$

When equality is reached, the complex eigenvalues collide and form a quartet of eigenvalues with nonzero real part



This signals an instability of the oscillations about the Lagrange points. The criterion for stability is

$$m = \frac{m_2}{m_1 + m_2} < 0.385$$

In the earth-moon system

$$m = \frac{m_{☾}}{m_{\oplus} + m_{☾}} = 0.012$$

and in the sun-Jupiter system

$$m = 0.95 \times 10^{-3}$$

so that in both cases the criterion for the stability of the Lagrange points is easily met.