Oct. 26

Perturbations of Periodic Motion

In the previous lecture, we studied the stability analysis of static solutions to the equations of mechanics. New issues arise when we consider the perturbation of systems in motion. The first context in which new problems arise is that of an oscillator with nonlinear coupling to other degrees of freedom. In this lecture, I will study this and related systems using straightforward analysis of the differential equations. Later in the course, I will return to this problem with more powerful methods derived from Hamiltonian mechanics.

Here is a first, simple example. Consider the nonlinear oscillator

$$\ddot{x} = -\omega^2 x - \lambda x^3$$

For an oscillation of amplitude A, the second term on the right is of size $\lambda A^2/\omega^2$ with respect to the first. Then, for small λ or small amplitude oscillations, we can regard the nonlinear term as a perturbation. We discussed this problem briefly in the first lecture. In particular, we noted there that the problem has a conserved energy

$$\frac{E}{m} = \frac{1}{2}(\dot{x})^2 + \frac{\omega^2 x^2}{2} + \frac{\lambda x^4}{4}$$

that regulates the general behavior of the oscillations.

Putting this insight aside for the moment, I will treat the differential equation by making an expansion of the solution in powers of λ,

$$x(t) = x_0(t) + \lambda x_1(t) + \lambda^2 x_2(t) + \cdots$$

Inserting this expression into the differential equation, we find a series of equations for the functions of t that are coefficients of the various powers of λ,

$$\lambda^0 \left(\ddot{x}_0 = -\omega^2 x_0 \right)$$
$$+ \quad \lambda^1 \left(\ddot{x}_1 = -\omega^2 x_1 - x_0^3 \right)$$
$$+ \quad \lambda^2 \left(\ddot{x}_2 = -\omega^2 x_2 - 3 x_0^2 x_1 \right) \quad + \ldots$$

The first equation is the linearized equation for $x(t)$, and we can immediately write the solution

$$x_0 = A \cos(\omega t + \phi)$$

I will set $\phi = 0$ for simplicity. From here on, each equation is an inhomogeneous linear equation for the new coefficient function $x_i(t)$. The solution to such an equation is built from a particular solution to the inhomogeneous equation, plus a general solution to the homogeneous equation. The homogeneous equation is that of the simple harmonic oscillator, so it seems that we can solve the successive equations in a straightforward way.

It is instructive to carry out the next step explicitly. The coefficient $x_1(t)$ obeys

$$\ddot{x}_1 + \omega^2 x_1 = -x_0^3 = -A^3 \cos^3 \omega t$$

The right-hand side can be simplified by

$$\cos^3 \omega t = \frac{\cos 3\omega t + 3 \cos \omega t}{4}$$

Consider first the perturbation induced by the $\cos 3\omega t$ term,

$$\ddot{x}_1 + \omega^2 x_1 = -\frac{A^3}{4} \cos 3\omega t$$

We can find a solution to the inhomogeneous equation of the form

$$x_1 = a \cos 3\omega t$$

The coefficient a is determined by

$$(-9\omega^2 + \omega^2)\, a = -\frac{A^3}{4}$$

so

$$a = \frac{A^3}{32\omega^2}$$

The other term in the perturbation, however, is more problemmatic. We need to solve

$$\ddot{x}_1 + \omega^2 x_1 = -\frac{3}{4} A^3 \cos \omega t$$

The obvious form

$$x_1 = b \cos \omega t$$

does not work, since it gives zero when substituted into the left-hand side. A little trial and error gives the solution

$$x_1 = -\frac{3}{8} \frac{A^3}{\omega} \cdot t \sin \omega t$$

The correction induced by this term has the awkward property that it grows arbitarily with time. Such an effect is called a *secular perturbation.* When secular terms appear, we cannot simply compute the successive terms in λ and add them as small perturbations. We need to use physical intuition to interpret the growing perturbation terms.

Secular terms arise when the perturbation, represented by a harmonic oscillator, is driven by a nonlinear term at the frequency of the oscillator. Another possible difficulty arises when the equation for a perturbation is driven by a nonlinear term at a frequency close to the frequency of the oscillator, for example,

$$\ddot{x}_1 + \omega^2 x_1 = -A^3 \cos \omega' t$$

where ω' is close to ω. The inhomogeneous solution to this equation is

$$x_1 = \frac{A^3}{\omega'^2 - \omega^2} \cos \omega' t$$

In this case, the size of the perturbation is bounded, but it becomes anomalously large as *omega'* approaches ω. This is called the problem of *small denominators.* The secular perturbation is the limit of this situation as $\omega' \to \omega$.

In either of these situations, we need to use physical arguments to understand how to treat the growing perturbation. The correction might correspond to a systematic growth or decay of the oscillation or to a change in the parameters of the oscillator. In systems with many degrees of freedom, more complex and interesting things can happen, as I will discuss later in the course.

In the particular case of the nonlinear oscillator that we are studying here, we know that the orbit cannot grow or decay, due to conservation of energy. I will now show that the secular term can be interpreted as a change in the frequency of the oscillation. To see this, try to solve the original equation using the ansatz

$$x(t) = A \cos\left((\omega_0 + \lambda\omega_1)t + \lambda x_1(t)\right)$$

Plugging this into the original differential equation, we find

$$\left[-(\omega_0+\lambda\omega_1)^2+\omega^2\right] A\cos(\omega_0+\lambda\omega_1)t \;+\; \lambda(\ddot{x}_1+\omega^2 x_1)$$

$$= \;-\lambda A^3\cos^3(\omega_0+\lambda\omega_1)t$$

$$= \;-\frac{3}{4}\lambda A^3\cos(\omega_0+\lambda\omega_1)t \;-\frac{1}{4}\lambda A^3\cos 3(\omega_0+\lambda\omega_1)t$$

We can satisfy this equation to order λ^0 by setting $\omega_0 = \omega$. Now look at the equation for the perturbation $x_1(t)$. This is

$$-2\lambda\omega\omega_1 \cdot A\cos(\omega_0+\lambda\omega_1)t \;+\; \lambda(\ddot{x}_1+\omega^2 x_1)$$

$$= \;-\frac{3}{4}\lambda A^3\cos(\omega_0+\lambda\omega_1)t \;-\frac{1}{4}\lambda A^3\cos 3(\omega_0+\lambda\omega_1)t$$

The perturbation proportional to $\cos(\omega_0+\lambda\omega_1)t$ appears on both sides of the equation. For the correct choice of ω_1, it cancels, so that there is no longer a secular perturbation. The condition for this is

$$\omega_1 = \frac{3}{8}\frac{A^2}{\omega}$$

Then the expansion for $x_1(t)$ contains only regular perturbations, and we can readily work out

$$x = A\cos\bar{\omega}t + \frac{\lambda A^3}{32\omega^2}\cos 3\bar{\omega}t$$

where

$$\bar{\omega} = \omega + \frac{3}{8}\lambda\frac{A^2}{\omega} + \cdots$$

The final form of the solution, to order λ, contains a modification of the form of the oscillation and a modification of the frequency. Continuing to higher orders, we can continue to adjust the frequency of the oscillation order by order in λ so that no secular terms appear.

We can generalize this discussion to systems with many degrees of freedom. In that context, there is a particular type of dangerous perturbation that I should explain and give a treatment for. With n degrees of freedom, the general solution depends on m initial conditions, and so is $2m$-dimensional. A perturbation might push the zeroth order solution from one of these functions to another one. The motion stays in the space of solutions, but departs from the original solution in a way that grows linearly with time.

This problem was already recognized by Lagrange and dealt with by the method of *variation of parameters*. To set up this method, rewrite the system of equations as a set of n first-order differential equations. Write these as

$$\dot{y}_i = f_i(y) + \lambda g_i(y)$$

with $i = 1, \ldots, n$. We regard the parameter λ as small. I will assume that we can find the complete solution of the unperturbed system

$$\dot{y}_i = f_i(y)$$

This depends on n initial conditions, or, equivalently, on n parameters c_i, $i = 1, \ldots, n$. Write this general solution as

$$y_i = Q_i(t, c)$$

Now add back the perturbation $\lambda g_i(y)$. I would like to look for an approximation solution to the full equation of a form similar to that in the example above. I will let the c_i take on a slow time-dependence and write

$$y_i = Q_i(t, c(t)) + \lambda z_i(t)$$

Then the equations, expanded to order λ take the form

$$\frac{\partial}{\partial t} Q_i + \dot{c}_j \frac{\partial}{\partial c_j} Q_i + \lambda \dot{z}_i$$

$$= f_i(Q) + \lambda g_i(Q) + \lambda \frac{\partial f_i}{\partial y_k} z_j$$

The Q_i obey the zeroth-order equation, so the terms of order λ^0 vanish. We would now like to use the variation of the c_i to soak up the slow variation in the solution induced by the perturbation, leaving any any rapid variation to the variations z_i. The c_i should then obey the equation

$$\dot{c}_j \frac{\partial}{\partial c_j} Q_i = \lambda g_i(Q)$$

where the right-hand side is evaluated at the current value of the c_i. This can be usefully written as a matrix equation. Introduce

$$c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \qquad g = \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} \qquad m_{ij} = \frac{\partial Q_i}{\partial c_j}$$

Then the equation becomes

$$m \dot{c} = \lambda g$$

or

$$\dot{c} = \partial \mathcal{M}^{-1} g$$

This formalism is illustrated by a problem from the theory of orbits. At the beginning of the course, we studied the equation for orbits in a central force field

$$\ddot{\vec{r}} = -\hat{r} f(r)$$

Using conservation of angular momentum, we rewrote this equation for motion in a plane as

$$\ddot{r} - \frac{c^2}{r^3} = -f(r)$$

where $c = L/m$, and then, changing variables to

$$\rho = \frac{1}{r} \qquad a' = \frac{d}{d\phi} a$$

we found

$$\rho'' + \rho = \frac{1}{\rho^2 c^2} f\left(\frac{1}{\rho}\right)$$

For an inverse square force

$$f(r) = \frac{g}{r^2}$$

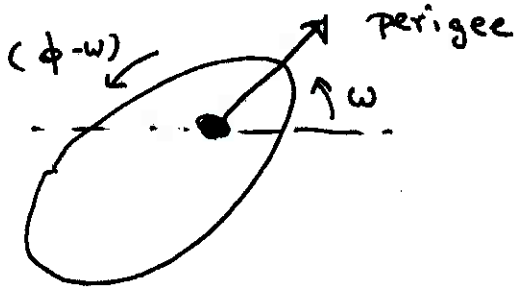
the equation becomes

$$\rho'' + \rho = \frac{g}{c^2}$$

The solution to this equation is

$$\rho = \frac{g}{c^2}\left(1 + e\cos(\phi - \omega)\right)$$

where e and ω are constants. Translating back to $r(\phi)$, this is an ellipse of eccentricity e



General relativity modifies gravity to produce a perturbation in the force law of the form

$$f(r) = \frac{g}{r^2} + \frac{\epsilon}{r^4}$$

I will now compute the effect of such a perturbation on the orbit.

The first step is to rewrite the orbit equation as a set of two first-order differential equations. Let $\sigma = \rho'$; then we find the system of equations

$$\rho' = \sigma$$

$$\sigma' + \rho = \frac{g}{c^2} + \frac{\epsilon}{c^2}\rho^2$$

The solution of the zeroth-order system is

$$\rho = \frac{a}{c^2}\left(1 + e\cos(\phi - \omega)\right)$$

$$\sigma = -\frac{a}{c^2} e \sin(\phi - \omega)$$

Now regard e and ω as slow functions of ϕ, varying with a speed of order ϵ. The equation $\rho' = \sigma$ becomes

$$-\frac{a}{c^2} e \sin(\phi-\omega) + \frac{a}{c^2} e' \cos(\phi-\omega) + \frac{a}{c^2} e\omega' \sin(\phi-\omega)$$

$$= -\frac{a}{c^2} e \sin(\phi-\omega)$$

The terms of zeroth order in ϵ cancel by construction, leaving a constraint between e' and ω'. We can expand the σ' equation similarly. The result is the system of equations

$$\frac{a}{c^2}\left(e'\cos f + e\omega' \sin f\right) = 0$$

$$\frac{a}{c^2}\left(-e'\sin f + e\omega'\cos f\right) = \frac{\epsilon}{c^2}\left(\frac{a}{c^2}\right)^2 (1 + e\cos f)^2$$

where I have written $f = \phi - \omega$. In matrix form, this reads

$$\begin{pmatrix} \cos f & \sin f \\ -\sin f & \cos f \end{pmatrix}\begin{pmatrix} e' \\ e\omega' \end{pmatrix} = \begin{pmatrix} 0 \\ \epsilon \frac{a}{c^4}(1+e\cos f)^2 \end{pmatrix}$$

or

$$\begin{pmatrix} e' \\ e\omega' \end{pmatrix} = \begin{pmatrix} \cos f & -\sin f \\ \sin f & \cos f \end{pmatrix}\begin{pmatrix} 0 \\ \frac{\epsilon a}{c^4}(1+e\cos f)^2 \end{pmatrix}$$

This system gives as the equation for e'

$$e' = -\frac{\epsilon g}{c^4} \sin f \, (1 + e \cos f)^2$$

Notice that the right-hand side of this equation integrates to zero around a cycle. Thus, if we average over time much longer than one orbit, the eccentricity of the orbit remains constant. Then we can treat e as a constant and study the long-time behavior of the ω' equation. This is

$$\omega' = \frac{\epsilon g}{c^4 e} \cos f \, (1 + e \cos f)^2$$

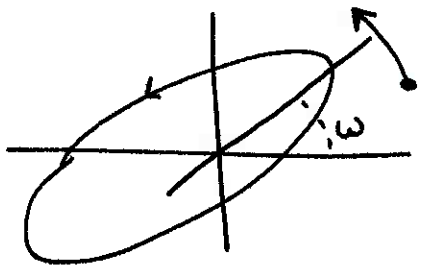
Over a cycle, ω changes by

$$\Delta\omega = \frac{\epsilon g}{e c^4} \int_0^{2\pi} df \, \cos f \, (1 + e \cos f) = \frac{\epsilon g}{e c^4} \cdot 2\pi e$$

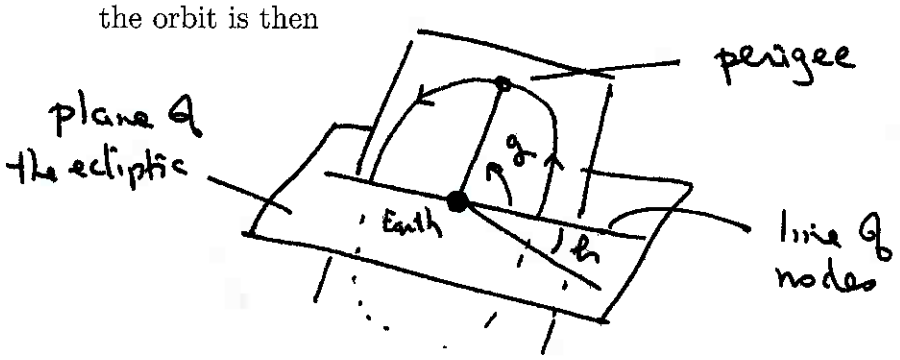
so that

$$\frac{\Delta\omega}{\text{cycle}} = 2\pi \frac{\epsilon g}{c^4}$$

The perihelion of the orbit precesses slowly in the forward direction



The theory of the earth-moon system contains several such slow variations of the orbit. To a first approximation, we may ignore the influence of the sun and other third bodies and regard the earth and moon as point masses. Then the relative coordinate $\vec{r}$ from the earth to the moon follows an ellipse. Since the Greeks, it has been known that the parameters of this ellipse vary as follows: The moon moves in a plane at an angle to the plane of the earth's orbit around the sun, called the *plane of the ecliptic.* This plane intersects the plane of the ecliptic in a line, called the *line of nodes.* The direction of this line is referred to a point among the fixed stars. The geometry of the orbit is then



The lunar orbit precesses in its plane, and also the orientation of the line of nodes precesses with respect to the fixed stars.

The two precession rates were first computed by Newton. Newton considered the effect of the sun, which exerts gravitational forces on both the earth and the moon. Since both the earth and the moon orbit the sun, the perturbation force that changes the moon's orbit is only the difference between the true force of the sun on the moon and the force that the moon would experience if it were at the position of the earth. Newton considered this as a small perturbation. He then derived the results

$$\frac{1}{n}\frac{d}{dt}(g+h) = \frac{3}{4}\left(\frac{n'}{n}\right)^2$$

$$\frac{1}{n}\frac{d}{dt}h = -\frac{3}{4}\left(\frac{n'}{n}\right)^2$$

where n is the orbital angular velocity of the moon around the earth and n' is the orbital angular velocity of the earth around the sun. The expansion parameter is then

$$\frac{n'}{n} = \frac{\text{month}}{\text{year}} \cong \frac{1}{12}$$

which would seem to be sufficiently small. A derivation of these results, by methods more modern and much easier than those used by Newton, is sketched in the next problem set.

Newton knew that his result for the precession of the line of nodes was in almost perfect agreement with the data. However, his prediction for the precession of the moon's perigee was too small by a factor of 2. Newton attributed this to observational error and literally hounded to death the Astronomer Royal, John Flamsteed, over this discrepancy.

Since Newton's time, a huge effort has been poured into attempts to understand this problem. The simplest explanation is that the higher-order terms in the series in n'/n turn out to be anomalously large, due to small denominators. In the 1700's, Clairaut and d'Alembert computed the next terms in the series and found

$$\frac{1}{n}\frac{d}{dt}(g+h) = \frac{3}{4}\left(\frac{n'}{n}\right)^2 + \frac{225}{32}\left(\frac{n'}{n}\right)^3 + \cdots$$

$$\frac{1}{n}\frac{d}{dt}h = -\frac{3}{4}\left(\frac{n'}{n}\right)^2 + \frac{9}{32}\left(\frac{n'}{n}\right)^3 + \cdots$$

The series for dh/dt still converges rapidly, but the series for $d(g+h)/dt$ is problemmatic. (The history of their interaction and priority dispute is described in a recent *Physics Today* article: Jan. 2010, p. 27.) In the nineteenth century, the American astronomer Hill introduced new methods, including particular exact solutions to the three-body problem, that gave better control over the series. In our own time, Gutzwiller and Schmidt computed thousands of terms in these series in order to understand chaotic variations in the very long time behavior of the lunar orbit.