Oct. 5

Noether's Theorem

In the previous lecture, I gave a rather practical introduction to the Lagrangian formulation of mechanics. I showed that a particle moving in a potential and some other systems of interest are described by the variational principle

$$\delta S[x] = 0 \quad \text{where} \quad S = \int_{t_1}^{t_2} dt \; L(x, \dot{x})$$

and the coordinate $x(t)$ is varied holding its initial and final values fixed. In this lecture, I will present Lagrangian mechanics in greater generality.

Newtonian mechanics is formulated in terms of coordinates x in an intertial, rectilinear frame. However, the statement $\delta S = 0$ is completely free of coordinates, and so we can evaluate it in any set of coordinates that we wish to use to parametrize the motion. In the last lecture, we saw that the variational approach is, for example, a very convenient way to derive equations of motion in polar coordinates.

We can work even more abstractly. Consider a mechanics problem with n dynamical variables or *degrees of freedom.* Choose any set of coordinates for this problem

$$q_i \qquad i = 1 \longrightarrow n$$

The Lagrangian for this problem will be a function of the q_i and $\dot{q}_i$

$$L(q_i, \dot{q}_i)$$

In this context, it is straightforward to work out the general variational equation of motion. Start from the action integral

$$S = \int_{t_1}^{t_2} dt \, L(q_i \, \dot{q}_i)$$

and vary the $q_i(t)$ under the condition

$$\delta q_i(t_1) = \delta q_i(t_2) = 0 \qquad \text{all } i$$

This gives

$$\delta S = \int_{t_1}^{t_2} dt \left\{ \delta q_i \frac{\partial L}{\partial q_i} + \delta \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right\}$$

$$= \delta q_i(t) \frac{\partial L}{\partial \dot{q}_i} \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} dt \, \delta q_i(t) \left\{ -\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{\partial L}{\partial q_i} \right\}$$

Here and always in the course, repeated indices should be understood to be summed over. The boundary term vanishes due to the condition that δq_i vanishes at the initial and final times. Then $\delta S = 0$ leads to the equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \qquad i = 1, \cdots, n \, .$$

These are called the *Euler-Lagrange equations.* One refers to the q_i as *generalized coordinates.* The quantity

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

is called the *generalized momentum* conjugate to q_i. With this notation, the Euler-Lagrange equations read

$$\frac{d}{dt} p_i = \frac{\partial L}{\partial q_i}$$

For a particle in a potential

$$L = \frac{1}{2} m (\dot{\vec{x}})^2 - V(\vec{x})$$

the momentum conjugate to x^i is

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = m \dot{x}_i$$

as we might have expected. The Euler-Lagrange equations are

$$\frac{d}{dt} (m \dot{x}_i) = - \frac{\partial}{\partial x_i} V$$

For a particle moving in a plane, we can change the description to polar coordinates. The Lagrangian now takes the form

$$L = \frac{1}{2} [m \dot{r}^2 + m r^2 (\dot{\phi})^2] - V$$

The generalized coordinates are r and ϕ. The corresponding conjugate momenta are

$$p_r = \frac{\partial L}{\partial \dot{r}} = m \dot{r} \qquad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m r^2 \dot{\phi}$$

Notice that p_ϕ is the *angular momentum*, which is conserved if the potential is purely a function of r. This latter statement follows from the Euler-Lagrange equation for ϕ, which reads

$$\frac{d}{dt} p_\phi = 0 \quad \text{for} \quad V = V(r) \text{ only}$$

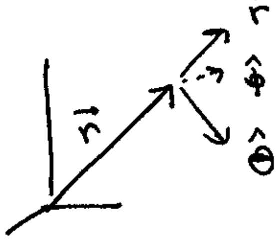
A conjugate momentum need not have units of momentum (g cm/sec). What is necessary instead is that the momentum p_i conjugate to q_i have units such that

$$p_i \cdot q_i \sim \text{g cm}^2/\text{sec} = \text{units of } \underline{\text{action}} \; S$$

In the example just considered, ϕ is dimensionless and p_ϕ has the units g cm^2/sec.

A particle moving in 3 dimensions, described in polar coordinates, has the Lagrangian

$$L = \frac{1}{2} m \left(\dot{r}^2 + (r\dot{\theta})^2 + (r \sin\theta \, \dot{\phi})^2 \right) - V$$



The three conjugate momenta are

$$p_r = m\dot{r}$$
$$p_\theta = m r^2 \dot{\theta}$$
$$p_\phi = m r^2 \sin^2\theta \, \dot{\phi}$$

Here

$$p_\phi = L_z$$

is the angular momentum about the $\hat{z}$ axis. If V is independent of ϕ, p_ϕ is a conservation law. If V is independent of all angles, the choice of the $\hat{z}$ axis is arbitrary, and so all three components of $\vec{L}$ must be conserved. This can be seen more explicitly in the following way: If V is independent of all angles, the equation of motion for p_θ is

$$\frac{d}{dt} p_\theta = r^2 \sin\theta \cos\theta \, (\dot{\phi})^2$$

$\dot{\vec{L}}$ is perpendicular to $\hat{r}$, and so only two equations are needed to verify the conservation of $\vec{L}$. If we compute $\dot{\vec{L}}$ explicitly in polar coordinates, the p_ϕ equation sets to zero the term proportional to $\hat{\theta}$, and the p_θ equation sets to zero the term proportional to $\hat{\phi}$.

To prepare the ground for the next topic, I will work out some formal properties of the action integral S. If we solve the Euler-Lagrange equations and then evaluate S on the extremal trajectory $q_i(t)$ that satisfies these equations subject to

$$q_i(0) = q_i \qquad q_i(T) = q_i' \qquad \overset{\text{sets}}{t_1 = 0 \quad t_2 = T}$$

we arrive at the value of the action

$$S[q_i, q_i' ; T]$$

How does this object depend on its arguments?

If we carry out a completely general variation of $q_i(t)$, we obtain the variation of S

$$\delta S = \delta q_i(T) \frac{\partial L}{\partial \dot{q}_i}(T) - \delta q_i(0) \frac{\partial L}{\partial \dot{q}_i}(0) + \int_0^T dt \, \delta q_i(t) (EL)_i$$

where $(EL)_i$ is the Euler-Lagrange equation. If we evaluate S on the path satisfying the Euler-Lagrange equations, the last term vanishes. Then

$$\frac{\partial S}{\partial q_i'} = \frac{\partial L}{\partial \dot{q}_i}(T) = p_i'$$

and, similarly,

$$\frac{\partial S}{\partial q_i} = -\frac{\partial L}{\partial \dot{q}_i}(0) = -p_i$$

We can also compute

$$\frac{\partial S}{\partial T} = \lim_{\Delta T \to 0} \frac{\Delta S}{\Delta T}$$

with q_i, q_i' fixed. We find

$$S[q_i\ q_i'; T+\Delta T] = S[q_i, q_i' - \Delta T \dot{q}_i', T] + \Delta T\, L(q_i', \dot{q}_i') + O(\Delta T^2)$$

Note that the first integral on the right-hand side involves a path from q_i to $q_i' - \Delta T \dot{q}_i'$. Then

$$\Delta S = -\Delta T \frac{\partial S}{\partial q_i'} \dot{q}_i' + \Delta T\, L(q_i', \dot{q}_i')$$

Finally,

$$\frac{\partial S}{\partial T} = -\left(p_i' \dot{q}_i' - L(q_i', \dot{q}_i') \right)$$

We will make use of this interesting formula later in the lecture.

One sometimes needs to evaluate the second derivative of S with respect to the endpoints of the path

$$\frac{\partial^2 S}{\partial q_i \, \partial q_i'}$$

For example, in the $\hbar \to 0$ limit of quantum mechanics, the amplitude for a particle to propagate from q_i to q_i' in time T is given by

$$(\text{const}) \cdot \left[\det \left(\frac{\partial^2 S}{\partial q_i \, \partial q_j'} \right) \right]^{1/2} e^{\frac{i}{\hbar} S(q, q', T)}$$

The phase is the Feynman phase factor discussed in the previous lecture. The prefactor is called the *van Vleck determinant.* Using the results above, we can evaluate the matrix contained in this factor as

$$\frac{\partial^2 S}{\partial q_i \, \partial q_j'} = \frac{\partial p_j'}{\partial q_i} = -\frac{\partial p_i}{\partial q_j'}$$

The equality of these derivatives is reminiscent of the Maxwell equations of thermodynamics.

Now we are ready to discuss one of the most beautiful results in physics, the theorem of Emmy Noether:

> For each continuously generated symmetry of the equation of motion, there is an associated conservation law.

By a *symmetry of the equations of motion*, I mean a transformation of the coordinates

$$q_i \to q_i' = F_i(q_j)$$

that leaves the Euler-Lagrange equations invariant. (Note that q_i' is being used here in a new sense, as the transformed version of the variable q_i.) By *continuously generated*, I mean that this transformation can be built up from infinitesimal variations.

$$q_i \to q_i' \simeq q_i + \alpha f_i(q) + \mathcal{O}(\alpha^2) \quad \text{or} \quad \delta q_i = \alpha f_i(q)$$

A concrete example is a rotation about the $\hat{z}$ axis

$$\begin{aligned} x &\to x' = x\cos\theta + y\sin\theta \\ y &\to y' = -x\sin\theta + y\cos\theta \end{aligned}$$

for which the infinitesimal transformation is

$$\begin{aligned} \delta x &= \alpha y \\ \delta y &= -\alpha x \end{aligned}$$

The finite transformation through θ can be thought of as built up as the product of N transformations through small angles α/N, which become infinitesimal in the limit $N \to \infty$. It is easy to find a Lagrangian such that L itself is left invariant by this transformation. Consider, for example,

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - V([x^2+y^2]^{\frac{1}{2}})$$

If the transformation is infinitesimally generated, it suffices to check that L is invariant under the infinitesimal transformation, including terms up to order α. For the Lagrangian just given,

$$\delta L = m\dot{x}\,\delta\dot{x} + m\dot{y}\,\delta\dot{y} - \delta x \frac{\partial}{\partial x} V - \delta y \frac{\partial V}{\partial y}$$

$$= m\dot{x}\,(\alpha\dot{y}) + m\,\dot{y}\,(-\alpha\dot{x}) - \left(\alpha y \frac{x}{r} - \alpha x \frac{y}{r}\right) \frac{\partial V}{\partial r}$$

Since the linear terms in α cancel, L is unchanged by an infinitesimal transformation, and thus the Euler-Lagrange equations must also be unchanged.

But, if L is invariant to the infinitesimal transformation parametrized by α, consider the result of varying $x(t)$ and $y(t)$ according to

$$\delta x = \delta\alpha(t) \cdot y$$

$$\delta y = -\delta\alpha(t) \cdot x$$

where now the parameter α is a function of t. If α is constant, there can be no change in L. Thus, for this more general variation, the variation of L must be of the form

$$\delta L = \delta\dot{\alpha} \cdot (\text{something})$$

Call the coefficient of $\delta\dot{\alpha}$ in this equation Q_α. Then the variation of the action is

$$\delta S = \int dt\ \delta\dot{\alpha}\ Q_\alpha = -\int dt\ \delta\alpha(t) \left(\frac{d}{dt} Q_\alpha\right)$$

Since S must be stationary under *all* variations of the q_i, we must have

$$\frac{d}{dt} Q_\alpha = 0$$

This is a conservation law. This proves a special case of Noether's Theorem: If there is an infinitesimally generated variation of the q_i that leaves the Lagrangian invariant, then there is an associated conservation law.

Actually, there are more general variations of the q_i that leave the Euler-Lagrange equations of motion invariant. If, for α constant

$$\delta L = \alpha \cdot \frac{d}{dt} F_\alpha$$

where F_α is a function of q_i and $\dot{q}_i$, then L changes under the variation but S is unchanged up to terms evaluated at the boundary. These terms do not affect the Euler-Lagrange equations of motion. So, such an α also gives an infinitesimally generated symmetry of the motion.

Now consider the effect of making an α that changes L in this way time-dependent. The variation of S is

$$\begin{aligned} \delta S &= \int dt \left(\delta\dot{\alpha}\, Q_\alpha + \delta\alpha \frac{d}{dt} F_\alpha \right) \\ &= -\int dt\, \delta\alpha(t) \left(\frac{d}{dt} Q_\alpha - \frac{d}{dt} F_\alpha \right) \end{aligned}$$

Then we see that

$$\frac{d}{dt} \left(Q_\alpha - F_\alpha \right) = 0$$

and again we find a conservation law. This completes the proof of Noether's Theorem.

Noether's Theorem not only proves the existence of these conservation laws but also shows us how to construct the conserved quantities. First, we construct Q_α. For a general infinitesimal transformation parametrized by α,

$$\delta q_i = \delta\alpha\, \phi_i(q)$$

$$\delta \dot{q}_i = \delta\alpha \, \dot{\phi}_i + \delta\dot{\alpha} \, \phi_i$$

Since L is a function of the q_i and $\dot{q}_i$, the last term here is the only possible source of a $\delta\dot{\alpha}$ term in the variation of the Lagrangian. If the Lagrangian is invariant to variations with constant α, then when α is time-dependent,

$$\delta L = \delta\dot{\alpha}_i \, \phi_i \frac{\partial L}{\partial \dot{q}_i} + \delta\alpha \frac{d}{dt} F_\alpha$$

We then recognize

$$Q_\alpha = \frac{\partial L}{\partial \dot{q}_i} \phi_i(q) = p_i \, \phi_i(q)$$

To include the case in which L changes by a total derivative, we add the term F_α. Then Noether's conservation law is given explicitly as

$$p_i \, \phi_i(q) - F_\alpha$$

I will now present a number of examples. First, consider the example studied above

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - V([x^2 + y^2]^{\frac{1}{2}})$$

For this system

$$\delta x = \alpha y \qquad \delta y = -\alpha x \qquad F_\alpha = 0$$

$$p_x = m\dot{x} \qquad p_y = m\dot{y}$$

so we can immediately write the conserved quantity as

$$Q_\alpha = (m\dot{x})\, y + (m\dot{y})(-x)$$

$$= -m\left[\, x\dot{y} - y\dot{x}\,\right]$$

This is just $(-L_z)$, the $\hat{z}$ component of angular momentum.

Next, consider a system of n coupled harmonic oscillators, all with the same frequency,

$$L = \frac{1}{2} m (\dot{x}_i)^2 - \frac{1}{2} m\omega^2 (x_i)^2$$

Intuitively, this system has the symmetry of rotations of a vector x_i in n dimensions. We will study rotational symmetry in more detail next week. For the moment, I will simply quote the form of the infinitesimal transformation

$$\delta x_i = a_{ij}\, x_j$$

where a_{ij} is an $n \times n$ real *antisymmetric* matrix. An antisymmetric $n \times n$ matrix has $n(n-1)/2$ independent components. These correspond to rotations in the $n(n-1)/2$ possible orthogonal planes through the origin in an n dimensional space. It is easy to see that this transformation leaves the Lagrangian invariant. For example

$$\delta (x_i)^2 = 2x_i \delta x_i = 2x_i a_{ij} x_j = 0$$

and similarly for the $(\dot{x}_i)^2$ term. The associated conservation laws are

$$\frac{d}{dt} Q = 0 \quad \text{for} \quad Q = m\dot{x}_i a_{ij} x_j$$

or, more clearly,

$$Q = - m a_{ij} x_i \dot{x}_j$$

The conserved quantities here are a sort of generalized angular momenta.

Next, consider the system discussed in the second lecture of the course, a system of particles interacting through central forces. The Lagrangian is

$$L = \sum_i \frac{1}{2} m_i (\dot{\vec{r}}_i)^2 - \sum_{(ij)} V_{(ij)}(|\overrightarrow{r_i - r_j}|)$$

This system is invariant under translations and rotations of the whole system of particles. We can easily work out the corresponding conservation laws.

Consider first the translations. The infinitesimal motions are

$$\vec{r}_i \to \vec{r}_i\,' = \vec{r}_i + \vec{a} \quad \text{or} \quad \delta\vec{r}_i = \vec{a}$$

Since

$$\dot{\vec{r}}_i \longrightarrow \dot{\vec{r}}_i$$

$$(\vec{r}_i - \vec{r}_j) \longrightarrow (\vec{r}_i - \vec{r}_j)$$

L is unchanged by this transformation. The correponding conservation law is proportional to

$$\sum_i \vec{p}_i \cdot \vec{a}$$

where

$$\vec{p}_i = \frac{\partial L}{\partial \dot{\vec{r}}_i} = m_i \dot{\vec{r}}_i$$

We recognize this as the conservation of total linear momentum,

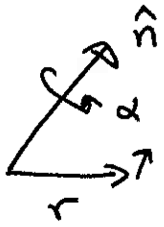
$$\vec{P} = \sum_i m_i \dot{\vec{r}}_i$$

The infinitesimal form of a rotation about the axis $\hat{n}$ is

$$\vec{r}_i \rightarrow \vec{r}_i + \alpha \hat{n} \times \vec{r}_i$$

or

$$\delta \vec{r}_i = \alpha \hat{n} \times \vec{r}_i$$



This transformation preserves the lengths of vectors. For example,

$$\delta(\dot{\vec{r}}_i)^2 = 2\dot{\vec{r}}_i \cdot (\alpha\, \hat{n}\times\dot{\vec{r}}_i) = 0$$

The term linear in α vanishes because $\hat{n} \times \dot{\vec{r}}$ is orthogonal to $\dot{\vec{r}}$. Thus, this transformation also leaves L unchanged. The conserved quantities are proportional to

$$\sum_i \vec{p}_i \cdot (\hat{n} \times \vec{r}_i)$$

Since

$$\vec{p}_i \cdot (\hat{n} \times \vec{r}_i) = (\hat{n} \times \vec{r}_i) \cdot \vec{p}_i = \hat{n} \cdot (\vec{r}_i \times \vec{p}_i)$$

we can recognize this as proportional to the total angular momentum

$$\vec{L} = \sum_i m_i\, \vec{r}_i \times \dot{\vec{r}}_i$$

Another well-known symmetry of a collection of nonrelativistic particles is the invariance under boosts between inertial frames of references, called *Galilean transformations*,

$$\vec{r}_i \rightarrow \vec{r}_i{}' = \vec{r}_i + \vec{V}t$$

Under this transformation,

$$(\vec{r}_i - \vec{r}_j) \to (\vec{r}_i - \vec{r}_j) \quad \text{so} \quad \delta V = 0$$

$$\dot{\vec{r}}_i \to \dot{\vec{r}}_i + \vec{V}$$

so

$$\delta\left(\frac{1}{2} m_i (\dot{\vec{r}}_i)^2\right) = m_i \dot{\vec{r}}_i \cdot \vec{V}$$

This implies that L transforms as

$$\delta L = \sum_i m_i \dot{\vec{r}}_i \cdot \vec{V} = \frac{d}{dt}\left[\sum_i m_i \vec{r}_i\right] \cdot \vec{V}$$

Thus, a Galilean transformation is a good enough symmetry to imply the existence of a conservation law. The associated conservation law is proportional to

$$\sum_i \vec{p}_i \cdot \vec{V} t - \sum_i m \vec{r}_i \cdot \vec{V}$$

This is just the statement

$$\frac{d}{dt}\left(\sum_i m \dot{\vec{r}}_i \cdot t - \sum_i m \vec{r}_i\right) = 0$$

This peculiar conservation law is equivalent to the constant motion of the center of mass

$$\sum_i m_i \ddot{\vec{r}}_i = 0 \quad \text{or} \quad M\ddot{\vec{R}} = 0$$

We have now derived almost all of the conservation alws of our assemblage of particles from the fundament space-time symmetries of this system:

translation in space $\longleftrightarrow$ $\vec{P}$ linear momentum

rotations in space $\longleftrightarrow$ $\vec{L}$ angular momentum

boosts $\longleftrightarrow$ $M\dot{\vec{R}}$ constant motion of the center of mass

Only one symmetry and one conservation law is missing, and it is easy to guess what it should be:

translation in <u>time</u> $\longleftrightarrow$ E energy

To check this, write the a time translation as the transformation

$$\vec{r}_i(t) \rightarrow \vec{r}_i{}'(t) = \vec{r}_i(t+\alpha)$$

The infinitesimal form is

$$\delta\vec{r}_i = \alpha\dot{\vec{r}}_i$$

When this is inserted into L, the Lagrangian changes by

$$\delta L = \alpha \frac{d}{dt} L(r, \dot{r})$$

Then the time translation is a Noether symmetry of the equations of motion. The associated conserved quantity is

$$\sum_i p_i \dot{\vec{r}}_i - L$$

Using the explicit form of the Lagrangian

$$L = \sum_i \frac{1}{2} m_i (\dot{\vec{r}}_i)^2 - V$$

we see that the conserved quantity is

$$\sum_i \frac{1}{2} m_i (\dot{\vec{r}}_i)^2 + V$$

which is the familiar form of the conserved energy.

The argument for energy conservation that I have just given can be generalized to *any* Lagrangian that does not depend explicitly on the time variable t. If L is a function of the q_i and $\dot{q}_i$ but does not depend on t except through its dependence on these variables, then the transformation

$$q_i \rightarrow q_i'(t) = q_i(t+\alpha) \quad \text{or} \quad \delta q_i = \alpha \dot{q}_i$$

causes L to transform as

$$\delta L = \alpha \frac{d}{dt} L$$

Then, from Noether's theorem, the following quantity is conserved:

$$p_i \dot{q}_i - L$$

It makes sense to call this the energy,

$$\frac{d}{dt} E = 0 \qquad \text{for} \qquad E = p_i \dot{q}_i - L$$

For a particle in a potential,

$$L = \frac{1}{2} m \dot{x}^2 - V$$

this gives the familiar result

$$E = \frac{1}{2} m \dot{x}^2 + V$$

We also recognize this quantity as the one that appeared in our earlier discussion of the derivatives of the action. We can now complete the set of formulae that we were constructing there. The full set of derivatives of action is

$$\left.\frac{\partial S}{\partial q_i'}\right|_{q_i, T} = p_i' \qquad \left.\frac{\partial S}{\partial q_i}\right|_{q_i', T} = -p_i \qquad \left.\frac{\partial S}{\partial T}\right|_{q_i q_i'} = -E$$

The generality of Noether's Theorem gives us a new view of the major conservation laws of physics. For the most general systems that are invariant under space-time symmetries, we can now *define energy* as the quantity conserved by virtue of time translation invariance, *momentum* as the quantity conserved by virtue of space translation invariance, and *angular momentum* as the quantity conserved by virtue of rotational invariance.

In addition to these conservation laws which follow from major space-time symmetries and are present under very general conditions, there are sometimes additional conservation laws that follow from non-intuitive symmetries. To see an example of this, I will return to the system of n harmonic oscillators discussed above.

$$L = \frac{1}{2} m(\dot{x}_i)^2 - \frac{1}{2} m\omega^2 (x_i)^2 \qquad i=1,\dots,n$$

We have already found $n(n-1)/2$ conservation laws of this system, but there are more. Consider the transformation

$$\delta x_i = t_{ij} \dot{x}_j$$

where t_{ij} is a *symmetric* matrix. Under this transformation,

$$\begin{aligned} \delta L &= m\dot{x}_i (t_{ij} \ddot{x}_j) - m\omega^2 x_i t_{ij} \dot{x}_j \\ &= \frac{d}{dt}\left[\frac{1}{2} m \, t_{ij} \dot{x}_i \dot{x}_j - \frac{1}{2} m\omega^2 t_{ij} x_i x_j \right] \end{aligned}$$

This transformation is a Noether symmetry and gives rise to the conserved quantities

$$\begin{aligned} & m\dot{x}_i t_{ij} \dot{x}_j - \left(\frac{1}{2} m \, t_{ij} \dot{x}_i \dot{x}_j - \frac{1}{2} m\omega^2 t_{ij} x_i x_j \right) \\ &= \left[\frac{1}{2} m \, t_{ij} \dot{x}_i \dot{x}_j + \frac{1}{2} m\omega^2 t_{ij} x_i x_j \right] \end{aligned}$$

Setting $t_{ij} = \delta_{ij}$, we find the conserved energy. However, taking t_{ij} to be another symmetric matrix, we find additional symmetries,

$$\frac{n(n+1)}{2} - 1$$

in all. Considering also the rotational symmetries discussed above, we have

$$\frac{n(n+1)}{2} - 1 \quad + \quad \frac{n(n-1)}{2} \quad = \quad n^2 - 1$$

symmetries in addition to time translation and the same number of conservation laws in addition to the total energy. For lovers of group theory, these symmetries generate the group $SU(n)$.

What are these symmetries good for? In classical mechanics, their role is very obscure. In quantum mechanics, however, the conserved quantities become operators that commute with the Hamiltonian. Applied to a state at a given energy level, they generate large multiplets of degenerate states. For the case $n = 3$, the lowest levels of the quantum harmonic oscillator have the degeneracies

$$\begin{array}{lcccc}
\frac{3}{2}\hbar\omega + 3\hbar\omega & — & a_i^{\dagger} a_j^{\dagger} a_k^{\dagger}|0\rangle & 10 & (7) + (3) \\
\frac{3}{2}\hbar\omega + 2\hbar\omega & — & a_i^{\dagger} a_j^{\dagger}|0\rangle & 6 & (5) + (1) \\
\frac{3}{2}\hbar\omega + \hbar\omega & — & a_i^{\dagger}|0\rangle & 3 & (3) \\
\frac{3}{2}\hbar\omega & — & |0\rangle & 1 & (1)
\end{array}$$

The numbers in parentheses give the degeneracies expected from 3-dimensional rotation symmetry alone; the additional degeneracies come from the new higher symmetries.