

Newtonian Mechanics

In this first lecture, I will review some basic formulae of Newtonian mechanics. I will also give examples, from the Newtonian viewpoint, of the qualitative analysis of dynamical systems. This will give us a first look at problems of perturbation theory and phase space motion that we will revisit from other points of view later in the course.

Newton described the motion of a particle in terms of a second order differential equation

$$m\ddot{x} = F(x, \dot{x}) \qquad \dot{x} = \frac{d}{dt}x$$

General theorems insure that, if we are given x \dot{x} at some time t_0 , then we can integrate this equation to determine $x(t)$ at later times as long as $F(x, \dot{x})$ remains finite.

A very simple example of this dynamical equation is the simple harmonic oscillator

$$F(x, \dot{x}) = -kx$$

Then

$$\ddot{x} = -\omega^2 x \qquad \omega^2 = \frac{k}{m}$$

You all know that the general solution of this equation is

$$x(t) = A \cos \omega t + B \sin \omega t$$

or

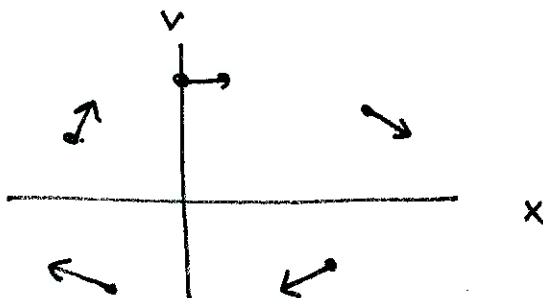
$$x(t) = A \cos \omega(t-T)$$

$$\dot{x}(t) = -A\omega \sin \omega(t-T)$$

I am sure that you can easily visualize this motion as a function $x(t)$. However, looking forward to more complicated cases, I would like to introduce a different method of visualization. If we can reduce the equation to a set of first-order differential equations, the motion becomes a simple flow through a higher-dimensional space. Rewrite Newton's equation, then, as

$$\dot{x} = v \qquad \dot{v} = \frac{1}{m} F(x, v)$$

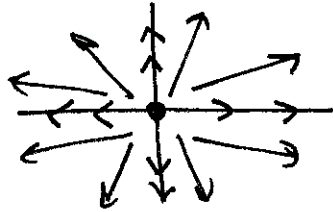
Now work in the (x, v) plane. This is called the *phase plane*. For each point in (x, v) , we can draw a vector proportional to (\dot{x}, \dot{v}) that gives the direction of motion in this plane.



Note that $v > 0$ implies $\dot{x} > 0$, so that the flow is to the right in the upper half plane and to the left in the lower half plane. Since each point in the phase plane has a specific flow vector associated with it, trajectories in the phase plane cannot cross. Trajectories can only meet at a point where the flow direction is ambiguous, that is, where $(\dot{x}, \dot{v}) = (0, 0)$. For example, the flow

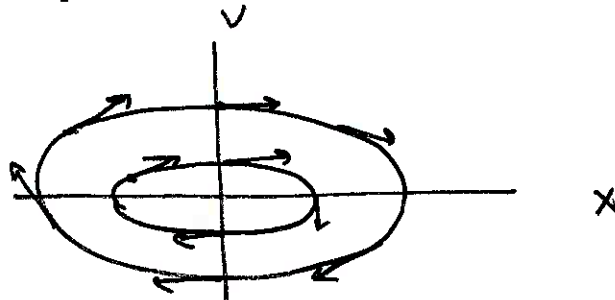
$$\dot{x} = a x \qquad \dot{v} = b v$$

leads to motion in the plane of the form



with trajectories meeting only at $(0,0)$.

For a simple harmonic oscillator, the motion is periodic. We must return repeatedly to the same point (x, v) from which we started. The trajectories in the phase plane must therefore form closed paths. From the equations above, we see that they are ellipses in the phase plane



The vectors (\dot{x}, \dot{v}) are tangent to the ellipses.

Any system with one coordinate obeying a second order differential equation can be analyzed in this way. Here are some further examples:

First, I will consider the *damped harmonic oscillator*. This system is described by the equation

$$F(x, \dot{x}) = -kx - \gamma \dot{x}$$

The phase plane equations are

$$\dot{x} = v \qquad \dot{v} = -\frac{\alpha}{m} v - \omega^2 x$$

This is a linear equation, and so we can solve it analytically by writing the solution as the real part of a complex expression

$$x = \operatorname{Re} [a e^{-i\Omega t}] \quad v = \operatorname{Re} [-i\Omega a e^{-i\Omega t}]$$

The equation then becomes

$$\operatorname{Re} (-\Omega^2 a = +i \frac{\gamma}{m} \Omega a - \omega^2 a) \cdot e^{-i\Omega t}$$

or

$$\Omega^2 + i \frac{\gamma}{m} \Omega - \omega^2 = 0$$

so that, finally,

$$\Omega = -i \frac{\gamma}{2m} \pm \left[\omega^2 - \frac{\gamma^2}{4m^2} \right]^{\frac{1}{2}}$$

From this, we can recognize the usual cases of damped harmonic motion. If γ is small, we have the *underdamped* case

$$x(t) = a e^{-\Gamma t} \cos \bar{\omega} t$$

with

$$\Gamma = \frac{\gamma}{2m} \quad \bar{\omega} = \left[\omega^2 - \frac{\gamma^2}{4m^2} \right]^{\frac{1}{2}}$$

If γ is large, we have *overdamped* motion

$$x(t) = a e^{-\Gamma_1 t} + b e^{-\Gamma_2 t}$$

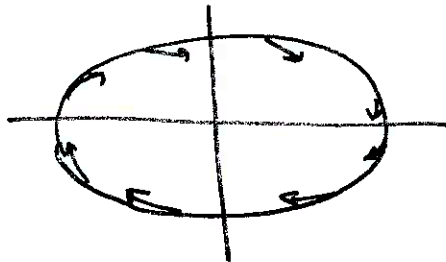
with

$$\Gamma_1, \Gamma_2 = \frac{\gamma}{2m} \pm \left[\frac{\gamma^2}{4m^2} - \omega^2 \right]^{1/2}$$

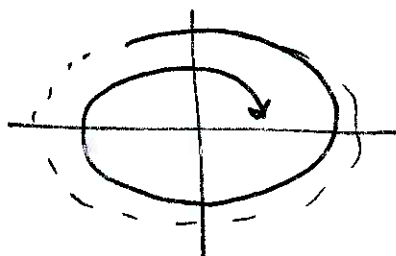
In the phase plane, the flow is given by the vector

$$(\dot{x}, \dot{v}) = (v, -\omega^2 x) - \left(0, \frac{\gamma}{m} v\right)$$

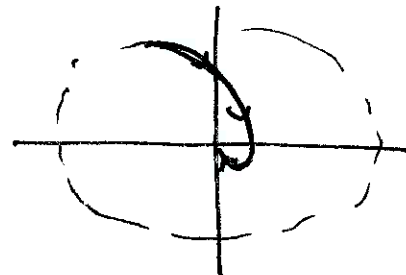
In the $\gamma = 0$ case, this vector is tangent to a closed ellipse. The term proportional to γ turns the vector so that it points into the ellipse of the $\gamma = 0$ motion



Then the trajectories spiral into the center of the phase plane. In the two cases, the trajectories have the form



underdamped



overdamped

If γ is small, the motion is approximately simple harmonic motion. We can analyze the motion by using the approximate form

$$x(t) \approx \bar{a}(t) \cos \bar{\omega} t$$

where $\bar{\omega}$ is a shifted frequency and $\bar{a}(t)$ has very slow or small dependence on t . This dependence should be proportional to some power of γ , and so we can consider $\dot{\bar{a}}$ and $\dot{\bar{\omega}}$ to be small parameters.

Working this out explicitly,

$$\dot{x} = v = -\bar{a} \bar{\omega} \sin \bar{\omega} t + \dot{\bar{a}} \cos \bar{\omega} t$$

$$\dot{v} = -\bar{a} \bar{\omega}^2 \cos \bar{\omega} t - 2 \dot{\bar{a}} \bar{\omega} \sin \bar{\omega} t + \dots$$

I have dropped a doubly small term with $\ddot{\bar{a}}$. The equation for \dot{v} is then

$$\begin{aligned} -\bar{a} \bar{\omega}^2 \cos \bar{\omega} t - 2 \dot{\bar{a}} \bar{\omega} \sin \bar{\omega} t \\ = -\omega^2 x - \frac{\gamma}{m} v \end{aligned}$$

To evaluate v on the right-hand side, we can take the expression for \dot{x} above. But, better, because v multiplies one power of the small parameter γ , we can approximate v by its leading-order expression. Then we find

$$\begin{aligned} -\bar{a} \bar{\omega}^2 \cos \bar{\omega} t - 2 \dot{\bar{a}} \bar{\omega} \sin \bar{\omega} t &= -\bar{a} \omega^2 \cos \bar{\omega} t \\ &+ \frac{\gamma}{m} \bar{a} \bar{\omega} \sin \bar{\omega} t \end{aligned}$$

Equating terms, we get a consistent first approximation

$$\bar{\omega}^2 = \omega^2 - 2\dot{\bar{a}}\bar{\omega} = + \frac{\gamma}{m} \bar{a}\bar{\omega}$$

This gives an evolution equation for a

$$\dot{\bar{a}} = - \frac{\gamma}{2m} \bar{a}$$

whose solution is

$$\bar{a}(t) = e^{-\Gamma t} \bar{a}(0) \quad \Gamma \hat{=} \frac{\gamma}{2m}$$

in agreement with the exact solution above. The corrections to $\bar{\omega}$ are of order γ^2 . We would find the first corrections by carrying out this analysis to one higher order.

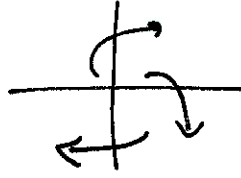
A more interesting example is the *van der Pol oscillator*,

$$F = -\gamma \left[\left(\frac{x}{b} \right)^2 - 1 \right] \dot{x} - m\omega^2 x$$

In this system, the damping is nonlinear. When $|x| \ll b$, the velocity-dependent force is

$$F \approx +\gamma \dot{x}$$

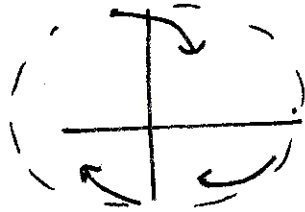
This is negative damping and leads to an increase in the amplitude of oscillations



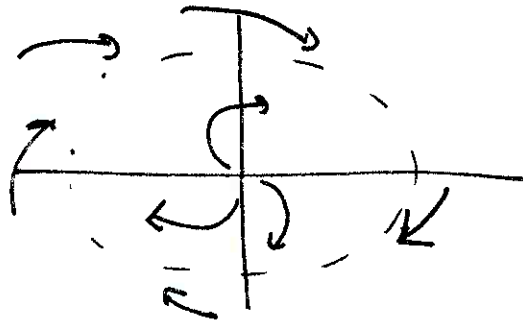
When $|x| \gg b$, the damping term is

$$F \approx -\gamma \left(\frac{x}{b}\right)^2 \dot{x}$$

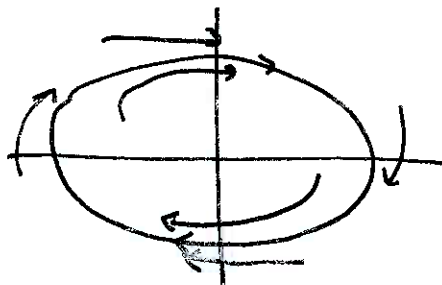
which leads to increasingly strong damping as $|x|$ increases.



Combining these features, the flows in the phase plane must have the form



The trajectories approach each other, but they cannot cross, as I have explained above. It must be, then, that all paths in the phase plane collapse onto a limiting trajectory that is periodic.



This limiting trajectory is called a *limit cycle*.

We can analyze the approach to the limit cycle by expanding in γ . Using our earlier equations, we have to first order in γ ,

$$\begin{aligned}\dot{v} &= -\bar{a}\bar{\omega}^2 \cos \bar{\omega}t - 2\dot{\bar{a}}\bar{\omega} \sin \bar{\omega}t \\ &= -\omega^2 x - \frac{\gamma}{m} \left(\left(\frac{x}{b} \right)^2 - 1 \right) v \\ &= -\bar{a}\omega^2 \cos \bar{\omega}t + \frac{\gamma}{m} \left[\frac{\bar{a}^2}{b^2} \cos^2 \bar{\omega}t - 1 \right] \bar{a}\bar{\omega} \sin \bar{\omega}t\end{aligned}$$

We can no longer exactly match coefficients. The nonlinearly damped motion contains not only the original harmonic oscillation, but also higher harmonics. However, we can try to find the best approximation of the form $\bar{a}(t) \cos \bar{\omega}t$. Multiply the equation for \dot{v} by $\cos \bar{\omega}t$ and integrate over a cycle. This projects out the $\cos \bar{\omega}$ terms on both sides of the equation and gives

$$-\bar{a}\bar{\omega}^2 = -\bar{a}\omega^2$$

This implies that

$$\bar{\omega} = \omega$$

up to corrections of order γ^2 . Now integrate both sides of the equation with $\sin \bar{\omega}t$. Since

$$\langle \sin^2 \bar{\omega}t \rangle = \frac{1}{2} \quad \langle \sin^2 \bar{\omega}t \cos^2 \bar{\omega}t \rangle = \frac{1}{8}$$

that gives the equation

$$\dot{\bar{a}} = -\frac{\gamma}{2m} \left(\frac{\bar{a}^2}{4b^2} - 1 \right) \bar{a}$$

This is an evolution equation for \bar{a} . You can see that \bar{a} increases if $\bar{a} < 2b$, and \bar{a} decreases if $\bar{a} > 2b$. To this order in perturbation theory, the limit cycle motion is

$$x = 2b \cos \omega t$$

If we linearize the equation for $\dot{\bar{a}}$

$$\bar{a} = 2b + \delta \bar{a}$$

$$\delta \dot{\bar{a}} = -\frac{\gamma}{2m} \cdot \frac{\delta \bar{a}}{b} \cdot 2b = -\frac{\gamma}{m} \delta \bar{a}$$

we see that neighboring trajectories collapse onto the limit cycle at the rate

$$\bar{a}(t) = 2b + \Delta e^{-\frac{\gamma}{m} t}$$

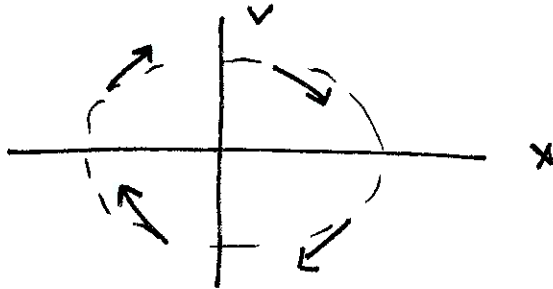
As a final example, I will consider the *nonlinear oscillator*

$$F = -m\omega^2 x - m\lambda x^3 \quad (\lambda > 0)$$

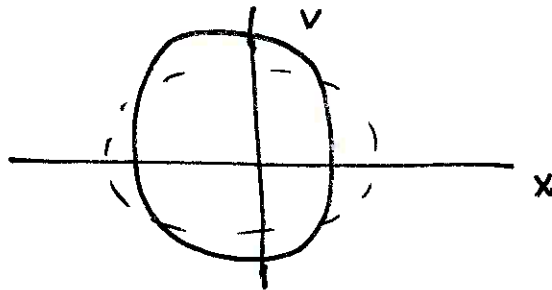
with $\lambda > 0$. For this system,

$$(\dot{x}, \dot{v}) = (v, -\omega^2 x - \lambda x^3)$$

The new term in \dot{v} shifts the flow vector, but in such a way that it sometimes enhances and sometimes retards the motion.



It seems that the new trajectory will have the shape



Now we would like to know: Is this a closed trajectory, giving a periodic motion at a fixed amplitude? We can first analyze this as we did the previous example, by looking for the best solution of the form

$$x = \bar{a}(t) \cos \bar{\omega} t$$

by working in perturbation theory in λ .

The \dot{v} equation is now

$$\begin{aligned} \dot{v} &= -\bar{a}\bar{\omega}^2 \cos \bar{\omega} t - 2\dot{\bar{a}}\bar{\omega} \sin \bar{\omega} t \\ &= -\omega^2 x - \lambda x^3 \\ &= -\bar{a}\omega^2 \cos \bar{\omega} t - \lambda \bar{a}^3 \cos^3 \bar{\omega} t \end{aligned}$$

All terms are even in $\bar{\omega} t$, so the integral with $\sin \bar{\omega} t$ will vanish. From the analysis of the previous case, that implies that

$$\dot{\bar{a}} = 0$$

at least to order λ . For the integral with $\cos \bar{\omega} t$, we need

$$\langle \cos^2 \bar{\omega} t \rangle = \frac{1}{2} \quad \langle \cos^4 \bar{\omega} t \rangle = \frac{3}{8}$$

Then, integrating with $\cos \bar{\omega} t$, we find

$$-\bar{a} \bar{\omega}^2 \cdot \frac{1}{2} = -\bar{a} \omega^2 \cdot \frac{1}{2} - \lambda \bar{a}^3 \cdot \frac{3}{8}$$

This equation implies

$$\bar{\omega}^2 = \omega^2 + \frac{3}{4} \lambda \bar{a}^2 + \dots$$

or

$$\bar{\omega} = \omega + \frac{3}{8} \frac{\lambda \bar{a}^2}{\omega} + \dots$$

To order λ , then, we find periodic motion at a shifted frequency that depends on the amplitude.

A periodic motion is one that neither increases nor decreases over time. A decreasing motion is *collapsing*, an increasing motion is *unstable*. A periodic motion at a constant amplitude might then we called a case of *marginal stability*. We should worry that, at any order of perturbation theory in λ , we might find a term that would render this motion unstable or cause it to collapse.

For this system, however, you all know that the motion is stable and periodic to all orders in λ . Go back to Newton's law,

$$\ddot{x} = -\omega^2 x - \lambda x^3$$

multiply by \dot{x} and integrate. We find

$$\frac{d}{dt} \left(\frac{1}{2} \dot{x}^2 + \frac{1}{2} \omega^2 x^2 + \frac{1}{4} \lambda x^4 \right) = 0$$

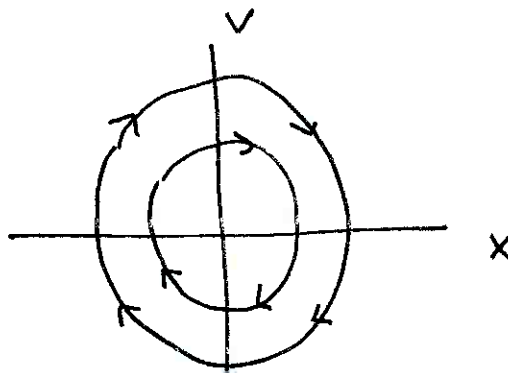
This is just the *conservation of energy*. In this more abstract language, we say that our oscillator has an *integral of the motion*

$$\frac{E}{m} = \frac{1}{2} v^2 + \frac{1}{2} \omega^2 x^2 + \frac{\lambda}{4} x^4$$

which is constant

$$\frac{d}{dt} E = 0$$

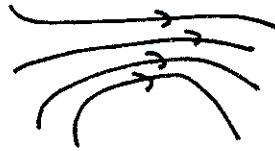
Then if we draw curves of constant E in the phase plane, the motion must be confined to a single curve.



The nested curves of constant E are the phase plane trajectories.

These simple oscillators exhibit many of the phenomena that we will study in more complicated systems. We will see more examples of stable and unstable motions, and stable oscillatory behavior. We will generalize the main idea of the phase plane approach, looking at dynamical systems as flows, constrained by the geometry of the space in which the flows take place. Hopefully, this will lead to a much richer set of tools with which to study dynamical systems.

I should point out an important complication found in more general systems. In two dimensions, the geometry of flows is highly constraining. If trajectories cannot cross, they must laminate the phase space smoothly



In three and higher dimensions, this is not necessary. We can have a disorderly arrangement of trajectories – chaos.



This idea of chaotic flows will also appear at several points in our study.