Sept. 30

Lagrangian Mechanics

In the previous lecture, I presented the calculus of variations and gave some examples of its application. In this lecture, I will explain the idea of *Lagrange* and *Euler* that the equations of motion for many mechanical system follow from a variational principle.

At the end of the last lecture, we studied a problem in which a free energy functional $F[m(z)]$ was to be minimized. We found that the variational equation gave the equation of motion of a particle mechanics system in which the potential energy was the *opposite* of what we expected. This motivates us to guess that the equation of motion of a general model of particle motion in a potential is given by a variational principle based on the functional

$$S[x(t)] = \int dt \{ T - V \}$$

The integrand $L = T - V$ is called the *Lagrangian.*

I will be more precise about this formulation. I describe the particle motion as a path $x(t)$ (or, more generally, $\vec{x}(t)$) beginning at x_0 at time t_0 and ending at position x_1 at time t_1. Let the *action integral* or *functional* $S[x(t)]$ be defined by

$$S(x_0, t_0, x_1, t_1 ; [x(t)]) = \int_{t_0}^{t_1} dt \left[\tfrac{1}{2} m \dot{x}^2 - V(x) \right]$$

I claim that the Newton equation for $x(t)$ follows from the variational principle

$$\delta S = 0$$

for variations of $x(t)$ keeping the initial and final positions fixed.

It is straightforward to check this explicitly. For simplicity of notation, take $t_0 = 0$, $t_1 = T$. Then

$$\delta S = \int_0^T dt \left\{ m\dot{x}\,\delta\dot{x} - \delta x \frac{\partial V}{\partial x} \right\}$$

$$= m\dot{x}\,\delta x \Big|_0^T + \int_0^T dt\, \delta x(t) \left\{ -m\ddot{x} - \frac{\partial V}{\partial x} \right\}$$

Since $\delta x = 0$ at $t = 0, T$, the surface term vanishes. Then $\delta S = 0$ for a general variation implies

$$m\ddot{x} = -\frac{\partial V}{\partial x}$$

I have given the argument in 1 dimension, but it generalizes directly to 3 dimensions and to any number of particles.

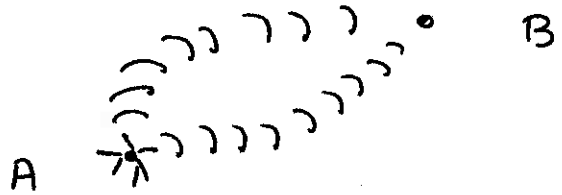
In the examples that we studied in the previous lecture, we used the calculus of variations to *minimize* the functional under study. In mechanics, however, we cannot ask for the action to be minimized. The kinetic energy T is positive and can take very large values in real particle motions. Similarly, V has large excursions, possibly of either sign. The solution to the equation $\delta S = 0$ are typically not either maxima or minima of $S[x]$. They are only *extrema*, points in the space where the first derivative with respect to a variation vanishes.

The idea that we need only *extremize* and not *minimize* $S[x]$ is peculiar and deserves some explanation. To begin, I will explain a point that was left hainging in the previous lecture: Why should the Fermat principle of least time be correct? To find the explanation, we have to realize that light propagates not as a particle but rather as a wave. The wave is emitted by an oscillating current, which we might take to have a definite frequency ω. The wave then propagates, accumulating the phase factor

$$e^{i\vec{k}\cdot\vec{x}}$$

in which $|\vec{k}| = \omega/v = \omega n/c$, where v is the phase velocity. If the medium has a varying index of refraction, the phase velocity will vary from place to place. We can

think of wave eminating from the source reflecting in various ways from interfaces and other scattering centers, and eventually making their way to the observer,



Waves moving along general paths arrive with varying phases

$$e^{i\int d\vec{x}\cdot\vec{k}(x)} = e^{i\omega\int dl\,\frac{n}{c}}$$

Waves with the same phase add coherently; waves with random phases cancel. The paths that contribute dominantly will then be paths such that nearby paths have the same phase and contribute coherently. This is exactly the principle that the time functional

$$T[x] = \int_A^B dl\,\frac{n(x)}{c}$$

be extremized.

The explanation for the use of a variational principle in particle mechanics is exactly the same. Lagrange was not aware that material particles like electrons and protons also propagate as waves, but now that is a well-known principle of physics. *Feynman* derived from the Schrödinger equation the statement that the motion of quantum mechanical particles can be described by waves following all possible paths, accumulating on each path the phase factor

$$e^{\frac{i}{\hbar}S(x_0,x_1;T)}$$

The dominant path is the one on which contributions from neighboring paths add coherently. This requires that the phase is extremized,

$$\delta S = 0$$

In the rest of this lecture, I will describe some simple applications of the variational formulation of particle mechanics. In the next lecture, we will consider a more general formulation, and more profound implications.

As a first example, consider particle motion in a plane about a fixed center of force. The Lagrangian is

$$L = T - V = \frac{1}{2} m(\dot{\vec{x}})^2 - V(x)$$

To find the equations of motion in polar coordinates, we might write Newton's laws and carefully convert the coordinate system. But it is easier just to rewrite the Lagrangian in polar coordinates and directly apply the equation $\delta S = 0$. The Lagrangian takes the form

$$L = \frac{1}{2} m[\dot{r}^2 + (r\dot{\phi})^2] - V(r)$$

Then

$$S = \int_0^T dt \left\{ \frac{1}{2} m(\dot{r})^2 + \frac{1}{2} m r^2 \dot{\phi}^2 - V(r) \right\}$$

and the variational equation is

$$\delta S = \int_0^T dt \left\{ m\dot{r}\,\delta\dot{r} + mr\,\delta r\,\dot{\phi}^2 + mr^2\dot{\phi}\,\delta\dot{\phi} - \delta r \frac{\partial V}{\partial r} \right\}$$

Integrating by parts and collecting terms, we have

$$\delta S = \int_0^T dt \left[\delta r \left\{ -m\ddot{r} + mr(\dot{\phi})^2 - \frac{\partial V}{\partial r} \right\} + \delta\phi \left\{ -\frac{d}{dt}(mr^2\dot{\phi}) \right\} \right]$$

We immediately recognize the variational equations as

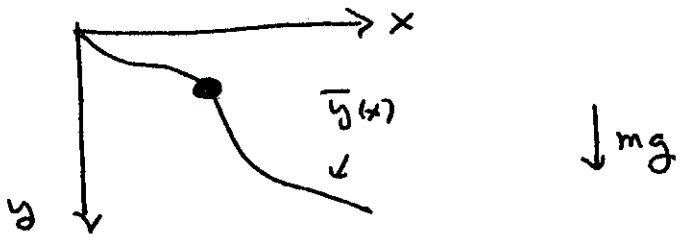
$$m\ddot{r} = mr(\dot{\phi})^2 - \frac{\partial V}{\partial r} \qquad \frac{d}{dt}(mr^2\dot{\phi}) = 0$$

The second of these equations is the conservation of angular momentum. The first equation is less familiar. However, you might remember that, in the previous lecture, we obtained after some analysis the equation

$$\ddot{r} - \frac{c^2}{r^3} = -\frac{g}{r^2}$$

The δr variational equation is exactly this for the special case in which $V(r)$ is a gravitational potential.

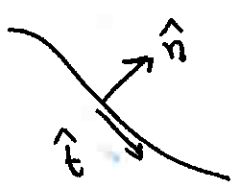
The fact that the variational principle is stated independently of the coordinate system makes it ideal for the analysis of constrained motion. Here is a sample problem: A particle is falling under gravity, but it is constrained to move on a fixed curve $\overline{y}(x)$. What is its equation of motion?

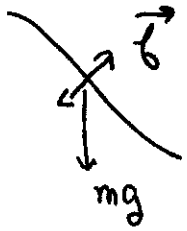


From the Newtonian point of view, this problem is not so easy. The equation of motion is

$$m\ddot{\vec{r}} = mg\hat{y} + \vec{f}$$

where $\vec{f}$ is the force that keeps the particle on the curve to which it is constrained. Let $\hat{t}$ be the unit vector tangent to the path and $\hat{n}$ be the unit vector normal to the path. The force $\vec{f}$ serves only to constrain the particle. It cannot accelerate the particle along the path or do any work on the particle. These considerations imply $\hat{t} \cdot \vec{f} = 0$, so the constraint force $\vec{f}$ must point normal to the path.





Then the equations of motion are

$$\hat{t} \cdot m\ddot{\vec{r}} = mg\,\hat{t} \cdot \hat{y} \qquad 0 = \hat{n} \cdot m\ddot{\vec{r}} = mg\,\hat{n} \cdot \hat{y} + \vec{f} \cdot \hat{n}$$

We can solve the $\hat{n}$ equation for the magnitude of $\vec{f}$ and then use this information to analyze the $\hat{t}$ equation.

Here is a similar variational treatment: Write the action as

$$S = \int dt \left\{ \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy + \lambda(y - \bar{y}(x)) \right\}$$

where the first term is the usual Lagrangian, with $V = -mgy$ for motion in a gravitational field. The second term introduces a new function $\lambda(t)$, called a *Lagrange multiplier*. The equations of the system are given by extremizing $S[x, y, \lambda]$ with respect to all three functions of t. The variation with respect to $\lambda(t)$ imposes the constraint

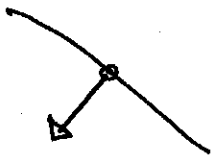
$$y = \bar{y}(x)$$

at every point. Then the motion must satisfy the constraint. The other two variational equations contain $\lambda(t)$.

$$m\ddot{y} = mg + \lambda(t)$$
$$m\ddot{x} = -\lambda(t) \cdot \frac{d\bar{y}}{dx}$$

The new term with λ appears as a force in the direction

$$-\frac{d\bar{y}}{dx}\hat{x} + \hat{y}$$



This force points in the direction $(-\hat{n})$ and is exactly the constraint force used in the previous method.

However, now that we see that the variational formulation of the problem is equivalent to the Newtonian one, we can change variables to make the problem easier to analyze. Let

$$y = \bar{y}(x) + \eta$$

introducing a new variable $\eta(t)$ to represent the motion parallel to $\hat{y}$ in violation of the constraint. The action becomes

$$S = \int dt \left\{ \frac{1}{2}m\left[\dot{x}^2 + \left(\frac{d\bar{y}}{dx}\dot{x} + \dot{\eta}\right)^2\right] + mg\,\bar{y}(x) + \lambda\eta \right\}$$

The equation of motion from varying the Lagrange multiplier is now

$$\eta(t) = 0$$

Then $\eta(t)$ must remain zero, and $\lambda(t)$ must be adjusted so that this is so.

The δx equation is worked out as follows:

$$\delta S = \int dt \left\{ m\dot{x}\delta\dot{x} + m\left(\frac{d\bar{y}}{dx}\right)^2 \dot{x}\delta\dot{x} + m\frac{d\bar{y}}{dx}\frac{d^2\bar{y}}{dx^2}\dot{x}^2\delta x + mg\frac{d\bar{y}}{dx}\delta x + \Theta(\eta)\right\}$$

$$= \int dt\ \delta x \left\{ -m\frac{d}{dt}\left[\dot{x}\left(1+\left(\frac{d\bar{y}}{dx}\right)^2\right)\right] + \frac{m}{2}\dot{x}\frac{d}{dt}\left(\frac{d\bar{y}}{dx}\right)^2 + mg\frac{d\bar{y}}{dx}\right\} + \cdots$$

If we set $\eta(t) = 0$, the Lagrange multiplier λ drops out of this equation. Simplifying, we find

$$m\ddot{x}\left(1+\left(\frac{d\bar{y}}{dx}\right)^2\right) + \frac{1}{2}m\dot{x}\frac{d}{dt}\left(\frac{d\bar{y}}{dx}\right)^2 - mg\frac{d\bar{y}}{dx} = 0$$

This is an equation for $x(t)$ that can be integrated directly.

To check that this is the correct equation, multiply by $\dot{x}$. That equation can be recognized as

$$\frac{d}{dt}\left[\frac{1}{2}m\dot{x}^2\left(1+\left(\frac{d\bar{y}}{dx}\right)^2\right) - mg\,\bar{y}(x)\right] = 0$$

This is just the conservation of energy for the particle falling along the curve to which it is constrained.

It is useful to review this theory of constrained motion a bit more formally. I consider general systems with n particle coordinates in which the constraints can be expressed as functions of the coordinates. We can then write the constraints explicitly in the form

$$\phi_i(x_1 \cdots x_n, t) = 0$$

for $i = 1, \ldots, c$, for c constraints. In this case, we say that the constraints are *holonomic*. For such systems, the Lagrangian

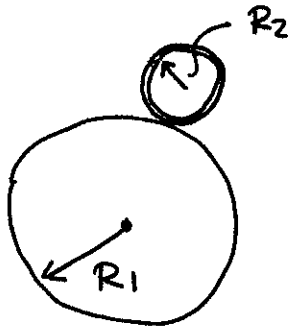
$$L = T - V + \sum_i \lambda_i(t)\, \phi_i(x_1 \cdots x_n, t)$$

gives an action integral that provides a correct description of the motion. If we can eliminate c variables to implement the constraints, we find $(n - c)$ equations of the form

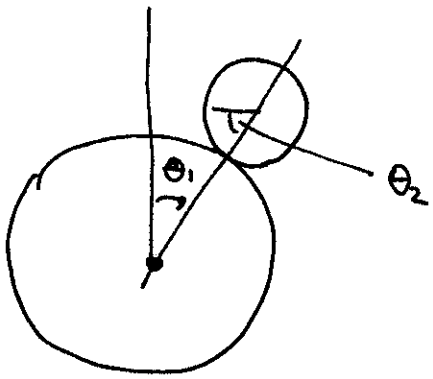
$$m\ddot{x}_j = -\frac{\partial}{\partial x_j} V$$

The remaining c equations of motion contain the λ_i and, essentially, are used to determine these functions, which are interpreted as constraint forces.

Here is one more example of this process, taken from Fetter and Walecka's book. Consider a cylindrical shell of mass m and radius R_2, positioned atop a fixed cylinder of radius R_1. If the upper cylinder rolls on the lower one without slipping, what is the motion?



The system has two coordinates, the angle of the upper cylinder and its point of contact. These are parametrized by angles θ_1, θ_2.



The condition of rolling without slipping is

$$R_1\theta_1 = R_2(\theta_2 - \theta_1)$$

This is the holonomic constraint. The Lagrangian describing the motion of the cylinder is

$$L = (\text{translational kinetic energy}) + (\text{rotational kinetic energy}) - (\text{gravitational potential}) + \lambda(R_1\theta_1 - R_2(\theta_2 - \theta_1))$$

or, more explicitly

$$L = \frac{1}{2}m[(R_1+R_2)(\dot{\theta}_1)]^2 + \frac{1}{2}m(R_2\dot{\theta}_2)^2 - mg(R_1+R_2)\cos\theta_1 - \lambda((R_1+R_2)\theta_1 - R_2\theta_2)$$

Use the constraint to eliminate θ_2,

$$L = \frac{1}{2}m[(R_1+R_2)\dot{\theta}_1]^2 + \frac{1}{2}m[(R_1+R_2)\dot{\theta}_1]^2 - mg(R_1+R_2)\cos\theta_1$$

then

$$\delta L = 2m(R_1+R_2)^2 \dot{\theta}_1 \delta\dot{\theta}_1 + mg(R_1+R_2)\sin\theta_1 \delta\theta_1$$

The variational equation for θ_1 is

$$2(R_1+R_2)\ddot{\theta}_1 = +g\sin\theta_1$$

which is the equation of an upside-down pendulum. This system has a conserved energy

$$E = m\left[(R_1+R_2)^2(\dot{\theta}_1)^2 + mg\cos\theta_1 (R_1+R_2)\right]$$

Now look at the constraint forces. The acceleration needed to keep an object on a circular path is

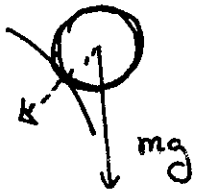
$$\frac{v^2}{r} \text{ or } r\dot{\theta}^2$$

which, for this system, is

$$(R_1+R_2)\dot{\theta}_1^2$$

This force must point in the radial (inward) direction. It must be supplied by the radial component of the gravitational force

$mg\cos\theta_1$



and a radial constraint force. This constraint force must equal

$$f_r = mg\cos\theta_1 - (R_1+R_2)\dot{\theta}_1^2$$

in the outward direction. If the cylinder starts at rest at $\theta_1 = 0$ and rolls down, we can determine $\dot{\theta}_1$ from conservation of energy and find

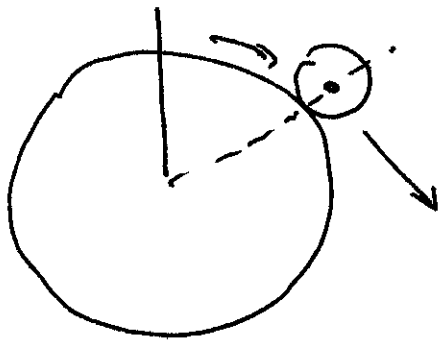
$$f_r = mg\cos\theta_1 - mg(1-\cos\theta_1)$$

$$= mg(2\cos\theta_1 - 1)$$

But, f_r must always be positive, directed outward. If the required f_r goes to zero, the upper shell is no longer being held on the lower cylinder, and it flies off. Apparently, this happens when

$$\cos\theta_1 = \frac{1}{2}$$

$$\theta_1 = \pi/3$$



So far, we have only discussed system of particles moving under potentials, in which the forces depend only on x and not on $v = \dot{x}$. We saw in the first lecture that v-dependent forces are often dissipative. We will see in the next lecture that this is inconsistent with a Lagrangian description, because the Lagrangian formulation leads, under rather general conditions, to system with conserved energy. Nevertheless, there are system with velocity-dependent forces that are described by Lagrangians. I would like to present a particularly important one, a particle moving through external electric and magnetic fields $\vec{E}(\vec{x},t)$, $\vec{B}(\vec{x},t)$. I use CGS notation.

I claim that this system is described by the following Lagrangian:

$$L = \frac{1}{2} m (\dot{\vec{x}})^2 + \frac{q}{c} \vec{A}(x,t) \cdot \dot{\vec{x}} - q \phi(x,t)$$

where $\phi(\vec{x}, t)$ is the scalar potential and $\vec{A}(\vec{x}, t)$ is the vector potential associated with $\vec{E}$ and $\vec{B}$. These fields are evaluated at the particle position $\vec{x}(t)$.

The variation of S is given by

$$\delta S = \int dt \left\{ m \dot{x}^i \delta \dot{x}^i + \frac{q}{c} A^i(x,t) \delta \dot{x}^i + \frac{q}{c} \delta x^i \frac{\partial A^j}{\partial x^i} \dot{x}^j - q \delta x^i \frac{\partial \phi}{\partial x^i} \right\}$$

Integrating by parts,

$$\delta S = \int dt \; \delta x^i \left\{ -m \ddot{x}^i - \frac{q}{c} \frac{d}{dt} \left[A^i(x(t),t) \right] + \frac{q}{c} \frac{\partial A^j}{\partial x^i} \dot{x}^j - q \frac{\partial \phi}{\partial x^i} \right\}$$

From this, we can read off the equation of motion

$$m \ddot{x}^i = -q \frac{\partial \phi}{\partial x^i} - \frac{q}{c} \left(\frac{\partial A^i}{\partial t} + \dot{x}^j \frac{\partial A^i}{\partial x^j} \right) + \frac{q}{c} \dot{x}^j \frac{\partial A^j}{\partial x^i}$$

We can recognize the appearance of the $\vec{E}$ field

$$E = -\frac{\partial}{\partial x^i} \phi - \frac{1}{c} \frac{\partial A^i}{\partial t}$$

and the $\vec{B}$ field

$$\frac{\partial}{\partial x^i} A^j - \frac{\partial}{\partial x^j} A^i = \epsilon^{ijk} B^k$$

With these identifications, we find for the equation of motion

$$m\ddot{x}^i = q E^i + \frac{q}{c} \epsilon^{ijk} \dot{x}^j B^k$$

that is

$$m\ddot{\vec{x}} = q \vec{E} + \frac{q}{c} \vec{v} \times \vec{B}$$

This is the familiar Lorentz force equation.

This system does not conserve energy and momentum, since the particle can exchange energy and momentum with the electromagnetic fields. However, if the dynamics of the fields $\phi(\vec{x}, t)$ and $\vec{A}(\vec{x}, t)$ is also described by a Lagrangian, this structure will guarantee that the total energy and momentum carried by particles and fields is conserved. I will give the argument to this point in the next lecture.