Nov. 18

The Transition to Chaos in Hamiltonian Systems

In the previous two lectures, we studied the transition to chaos in a system of ordinary differential equations and in a functional recursion. I will now take what we have learned up to the level of Hamiltonian systems. For Hamiltonian systems, we have the highly regulated structure of phase space flows described in previous lectures. In particular, we have argued that, in many circumstances, flows can be straightened out into the regular motion of angle and action variables. The main question that I will address in this lecture is, then, how can this structure break down in an interacting nonlinear system?

As a preliminary step in this discussion, I need one additional formula for Hamiltonian mechanics that was omitted in the previous lectures. Up to now, we have dealt only with fixed canonical transformations, but in this lecture, I will consider time-dependent canonical transformations. We will want to know how the Hamiltonian transforms under such a transformation.

There is a prima facie argument that the Hamiltonian is invariant to canonical transformations. Earlier in the course, we derived a formula for the Hamiltonian in terms of the action integral from the initial with the q_i fixed to the final conditions with the Q_i fixed, calculated on the path that extremizes the action. Calling this quantity

$$S(q, Q; T)$$

we saw that the Hamiltonian is given by

$$H = -\frac{\partial}{\partial T} S(q, Q; T)\Big|_{q,Q}$$

and is time-independent. This derivation makes no reference to a particular set of canonical variables. However, this derivation assumes that there is no explicit time-dependence in the dynamics, so to analyze the effect of a time-dependent transformation of variables, we need to be a little more careful.

Consider, then, two sets of phase space variables (q_i, p_i) and (Q_i, P_i) related by a, possibly time-dependent, canonical transformation. Let the original Hamiltonian be $H(q,p)$ and the transformed one be $K(Q,P)$. In terms of the first set of variables, the action integral is

$$S = \int dt \; [p_i \dot{q}_i - H(q,p)]$$

In the second set of variables, the action integral is, similarly,

$$S' = \int dt \; [P_i \dot{Q}_i - K(Q,P)]$$

These two action principles must describe the same set of motions, and so they must have the same extrema, determined by the condition $\delta S = 0$. Thus, S and S' can only differ by boundary terms. This means that the integrands can differ only by a total time derivative

$$(p_i \dot{q}_i - H) - (P_i \dot{Q}_i - K) = \frac{d}{dt} \mathcal{F}$$

The generating function $F_1(q, Q, t)$ satisfies

$$\begin{aligned} \frac{dF_1}{dt} &= \dot{q}_i \frac{\partial F_1}{\partial q_i} + \dot{Q}_i \frac{\partial F_1}{\partial Q_i} + \frac{\partial F_1}{\partial t} \\ &= \dot{q}_i p_i - \dot{Q}_i P_i + \frac{\partial F_1}{\partial t} \end{aligned}$$

In the case where F_1 has no explicit time-dependence, the relation for the two action integrands is satisfied with

$$\mathcal{F} = F_1(q,Q) \qquad K = H$$

In case where the generating function has explicit time-dependence, we can still set

$$\mathcal{F} = F_1(q, Q)$$

but now

$$K = H + \frac{\partial F_1}{\partial t}$$

The generating function $F_2(q, P, t)$ is related to $F_1(q, Q, t)$ by a time-independent Legendre transformation

$$F_2(q, P, t) = F_1(q, Q, t) + Q_i P_i$$

so it is also correct that

$$K = H + \frac{\partial F_2}{\partial t}$$

With this formula in hand, I will discuss the effect of nonlinear perturbations on a Hamiltonian system. I will assume that the Hamiltonian can be written as

$$\mathcal{H} = H_0 + \lambda V$$

where H_0 describes an integrable system with periodic solutions and λ is a small parameter. This Hamiltonian describes, for example, a set of particles bound in simple potentials with weak nonlinear couplings between them.

As a first step, convert the original variables to angle-action variables for H_0. In these coordinates, the Hamiltonian takes the form

$$H = h_0(J) + \lambda V(\phi, J).$$

Each coordinate executes uniform periodic motion at the frequency

$$\omega_i = \frac{\partial h_0}{\partial J_i}$$

The nonlinear perturbation drives each of these angle variables in a time-dependent way. For concreteness, you might think about the case of two oscillators coupled by the interaction

$$V = \lambda x_1^2 x_2^2$$

Consider a solution in which the first oscillator has frequency ω_1 and the second has frequency ω_2. If we plug the solution of the unperturbed equations into the extra term that comes from the perturbation, we see that the perturbation forces the first oscillator at the frequency $2\omega_2$. In general, the nonlinear perturbation will disturb the equation for ϕ_i at a frequency

$$\sum_j n_j \omega_j$$

In the leading order of perturbation theory, the n_j will be small integers. In higher orders of perturbation theory, higher powers of V will appear, and perturbations with higher integers n_j will be generated.

To analyze the effect of these perturbations, I will first consider the simpler problem of a single oscillator, with coordinates ϕ, J and Hamiltonian

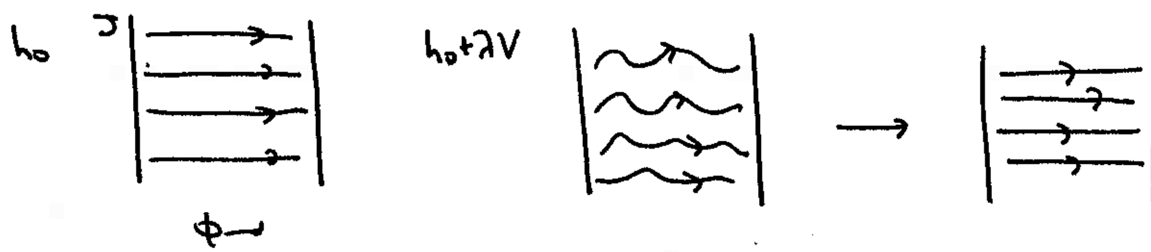
$$H = h_0(J) + \lambda V(\phi, J, t)$$

perturbed by a given time-dependent perturbation $V(\phi, J, t)$. The ϕ variable is periodic with period 2π, so we can represent V by a Fourier series

$$V(\phi, J, t) = \sum_{m\,\Omega} e^{im\phi} e^{-i\Omega t} V_{m,\Omega}(J)$$

I will assume that the term $\Omega = 0, m = 0$ is absent. This term is a constant in t and ϕ, so if V contains this term, we can immediately move it into $h_0(J)$.

The perturbation V causes the phase space flows to be distorted by an amount of order λ. In Hamiltonian mechanics, the natural response would be to return the flows to Hamilton-Jacobi form by making a canonical transformation.



I will now construct this transformation for a generic perturbation, removing the leading, order λ term in H that depends on ϕ and t. In principle, it should be possible to apply this set of manipulations repeatedly to convert H completely to Hamilton-Jacobi form.

I will write the canonical transformation using the generating function

$$F_2(\phi, \bar{J}, t) = \phi \bar{J} - \lambda G(\phi, \bar{J}, t)$$

where (ϕ, J) are the original coordinates and $(\bar{\phi}, \bar{J})$ are the new coordinates. This is an order λ perturbation of the identity. The old and new coordinates are related by

$$\bar{\phi} = \phi - \lambda \frac{\partial G}{\partial \bar{J}} \qquad J = \bar{J} - \lambda \frac{\partial G}{\partial \phi}$$

The new Hamiltonian is

$$\bar{H} = H - \lambda \frac{\partial G}{\partial t}$$

Thus

$$\bar{H} = h_0\left(\bar{J} - \lambda \frac{\partial G}{\partial \phi}\right) + \lambda V\left(\phi, \bar{J} - \lambda \frac{\partial G}{\partial \phi}, t\right) - \lambda \frac{\partial G}{\partial t}$$

Now

$$h_0\left(\bar{J} - \lambda \frac{\partial G}{\partial \phi}\right) = h_0(\bar{J}) - \lambda \underbrace{\omega(\bar{J})}_{= h_0'(\bar{J})} \frac{\partial G}{\partial \phi} + \cdots$$

up to an error of order λ^2. The term

$$\lambda V\left(\phi, \bar{J} - \lambda \frac{\partial G}{\partial \phi}, t\right) \approx \lambda V(\phi, \bar{J}, t)$$

is already of order λ, so we can neglect higher-order pertubations of this term. Then, to order λ, $\overline{H}$ takes the form

$$\bar{H} = h_0(\bar{J}) + \lambda \left[-\omega(\bar{J}) \frac{\partial G}{\partial \phi} + V(\phi, \bar{J}, t) - \frac{\partial G}{\partial t} \right] + \cdots$$

Then if we can arrange to cancel the term in brackets

$$\left[\omega(\bar{J})\frac{\partial}{\partial\phi} + \frac{\partial}{\partial t}\right] G = V(\phi, \bar{J}, t)$$

we will have returned the Hamiltonian to the form

$$\bar{H} = h_0(\bar{J})$$

up to terms of order λ^2. (Remember that the term $h_0(J)$ will have soaked up any term in V that is independent of ϕ and t.)

Now we have a definite equation that we can solve for $G(\phi, \bar{J}, t)$. In the Fourier representation, this equation becomes

$$(i m \omega(\bar{J}) - i\Omega)\, G_{m,\Omega} = V_{m,\Omega}(\bar{J})$$

and so

$$G(\phi, \bar{J}, t) = \sum_{m\,\Omega} e^{im\phi - i\Omega t} \left(\frac{-i}{m\,\omega(\bar{J}) - \Omega}\right) V_{m,\Omega}(\bar{J})$$

There is a serious problem with this solution, however. For a secular perturbation satisfying the condition

$$m\,\omega(\bar{J}) = \Omega$$

the Fourier coefficient in G is infinite. Even when the perturbation is close to this resonance, the response to the small perturbation is amplified by a small denominator.

I claimed above that it is possible to remove the effect of a generic perturbation by a canonical transformation. That is true in some technical sense, because the resonance condition is satisfied only for some particular conditions. However, the problem of resonances and small denominators cannot be ignored in a nonlinear system. First, as I have explained above, the frequency Ω is a linear combination of the frequencies of the coupled oscillators

$$\Omega = n_i \omega_i (J)$$

At high orders of perturbation theory, we can use large integers n_j and m and find values that approximate the resonance condition

$$m\omega = n_i \omega_i$$

increasingly closely. Second, unless the unperturbed system has perfect harmonic oscillators, the frequency

$$\omega(J)$$

will vary over a range is J. It is likely that, at some particular value $J = J_r$, the condition

$$m\omega(J_r) = \Omega$$

will be satisfied.

It is therefore important to analyze the case of a perturbation that is very close to resonance. As a model for this, consider the Hamiltonian

$$H = h_0(J) + \lambda f(J) \cos(m\phi - \Omega t)$$

considered for values of J in the neighborhood of a point J_r satisfying the resonance condition.

As a first step in analyzing this problem, we can simplifying it with the canonical transformation generated by

$$F_2 = \left(\phi - \frac{\Omega t}{m}\right) \bar{J}$$

Then

$$\bar{\phi} = \phi - \frac{\Omega t}{m} \qquad J = \bar{J}$$

The new Hamiltonian is

$$\bar{H} = H + \frac{\partial F_2}{\partial t} = h_0(\bar{J}) - \frac{\Omega \bar{J}}{m} + \lambda f(\bar{J}) \cos m\bar{\phi}$$

We can expand the terms in this new Hamiltonian in J near $J = J_r$,

$$f(J) = f(J_r) + f'(J_r)(J - J_r) + \cdots$$

$$h_0(J) = h_0(J_r) + \omega(J_r)(J - J_r) + \frac{1}{2}\omega'(J_r)(J - J_r)^2 + \cdots$$

Assembling $\overline{H}$, we find

$$\bar{H} = H_0(J_r) + \cancel{\omega(J_r)(J-J_r)} + \tfrac{1}{2}\omega'(J_r)(J-J_r)^2$$
$$- \frac{\Omega}{m} J_r - \cancel{\frac{\Omega}{m}(J-J_r)} + \lambda f(J_r) \cos m\phi + \cdots$$

Dropping terms that are simple constants, we find for the new Hamiltonian

$$\bar{H} = \frac{1}{2}\omega'(J_r)(J-J_r)^2 + \lambda f(J_r) \cos m\phi$$

This system is of the form of the Hamiltonian of a pendulum

$$H = \frac{J^2}{2I} + Mg \cos\phi$$

with the identifications

$$\omega'(J_r) = \frac{1}{I} \qquad\qquad \lambda f(J_r) = Mg$$

With the choice of signs above, the pendulum points *up*, at the unstable point of its potential, at

$$\phi = 0, \pm\frac{2\pi}{m}, \pm\frac{4\pi}{m}, \ldots$$

and *down*, at the stable point of its potential, at

$$\phi = \pm\frac{\pi}{m}, \pm\frac{3\pi}{m}, \ldots$$

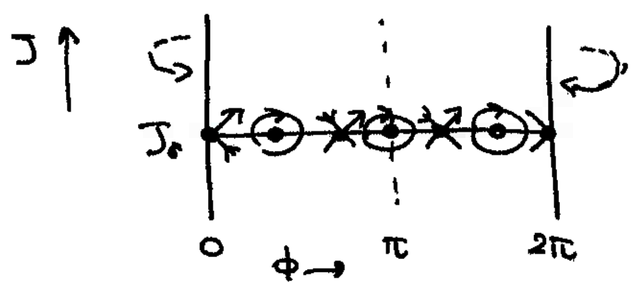
Then there are stable fixed points at

$$J = J_r \qquad \phi = \frac{2\pi(k+\frac{1}{2})}{m}, \qquad k = -2, -1, 0, 1, 2 \cdots$$

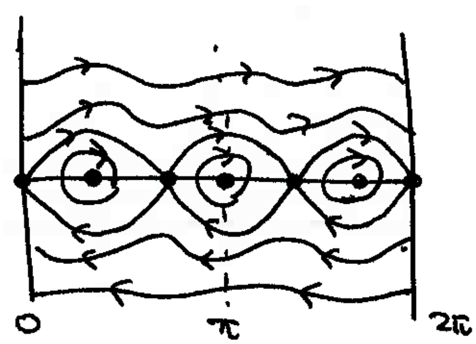
and unstable fixed points at

$$J = J_r \qquad \phi = \frac{2\pi k}{m} \qquad k = \cdots, -2, -1, 0, 1, 2, \cdots$$

In the literature on chaos in Hamiltonian system, these points are called, respectively, the *elliptic* and *hyperbolic cycles.* If I choose $m = 3$ for definiteness, the phase space flows have the form



where I have drawn closed orbits around the stable points and diverging trajectories from the unstable points. A more complete picture of the phase space flows is



The stable region around each stable fixed point is enclosed by a *separatrix* that isolates it from the overall periodic flow of ϕ.

This is a very interesting picture. For values of J away from the resonance condition, we have smooth periodic orbits just as in the unperturbed system. However, for values of J near J_r, there is an essential disruption of the phase space flows, with a new structure of fixed points. Here, the effect of the perturbation is literally a rip in the fabric of time!

You might recall from our discussion of the Hénon-Heiles system that the behavior on the Poincaré section should regions of smooth flow that we called *stable islands.* The whole region covered by our model system might be part of one such stable island. We see that a resonant perturbation disrupts this smooth motion. A very small perturbation generates new, smaller, stable islands in the regions near the elliptic points. However, these also can be disrupted at higher orders of perturbation theory.

It is interesting to ask, how large is the disrupted region? We can use the width of the separatrix to estimate this. To find the equation for the separatrix, we can use the fact that the Hamiltonian $\overline{H}$ is time-independent and therefore is conserved. Write again

$$\overline{H} = \frac{1}{2}\omega'(J_r)(J-J_r)^2 + \lambda f(J_r)\cos m\phi$$

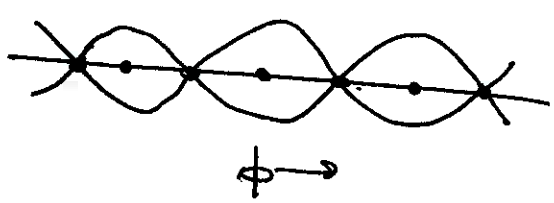
At the unstable fixed points, the value of the cosine is 1. Then, on the separatrix,

$$\overline{H} = \lambda f(J_r)$$

We can now solve for $J(\phi)$ along this trajectory; we find

$$J = J_r \pm \left[\frac{2\lambda f(J_r)}{\omega'(J_r)}(1-\cos m\phi)\right]^{1/2}$$

where $\pm$ describes the upper/lower separatrix,

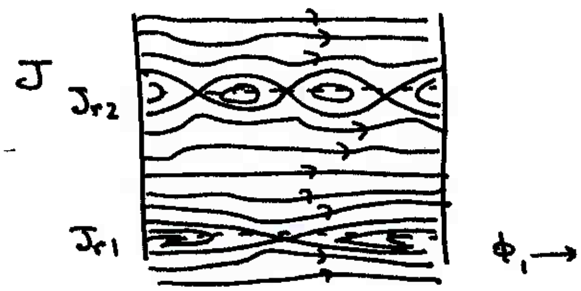


The size of the isolated regions around the stable fixed points is of order $\lambda^{1/2}$.

If the simple system we have just analyzed is a model for a nonlinear system with two degrees of freedom, we will see disruption of the phase space flow when any of the resonance conditions of the form

$$m\omega_1 = n\omega_2$$

are met. The leading effects, from low orders of perturbation theory, come from small integers n. If λ is small, the disrupted regions will be well separated along the J axis,



We will see regular motion in most of the phase space as long as λ is small.

Let

$$\Delta\left(\frac{n}{m}\right)$$

be the spacing of adjacent relevant values of the ratio of integers n/m. The resonance condition is

$$\omega_1(J_r) = \frac{n}{m}\,\omega_2$$

so the spacing of resonances in J is given by

$$\Delta J_r = \frac{\omega_2}{\omega_1'(J_r)} \cdot \Delta\left(\frac{n}{m}\right)$$

On the other hand, the size of the disrupted region of phase space is

$$\delta J = 2 \cdot \left[\frac{4\lambda f(J_r)}{\omega'(J_r)} \right]^{\frac{1}{2}}$$

So neighboring resonances are well separated in J if

$$2 \cdot 2 \cdot \left[\lambda \frac{f(J_r)}{\omega'(J_r)} \right]^{\frac{1}{2}} < \frac{\omega_2}{\omega'(J_r)} \Delta\left(\frac{n}{m}\right)$$

or

$$4 \left[\lambda f(J_r)\, \omega'(J_r) \right]^{\frac{1}{2}} < \Delta\left(\frac{n}{m}\right)$$

This is the *Chirikov criterion.* When this criterion is satisfied, we can avoid resonances and find operating points for the system for which the degrees of freedom evolve as simply as in the unperturbed case. Clearly, it is easier to satisfy the Chirikov condition when λ is small. It is also important that $\omega'(J_r)$ not be too large, that is, that the original oscillators not be too anharmonic. On the other hand, if the original oscillators are almost perfectly harmonic and $\omega'(J_r)$ is very small, the region disrupted by each resonance will be very large. A little nonlinearity can thus be a good thing. In the design of a mechanical system it is necessary to balance these two effects.

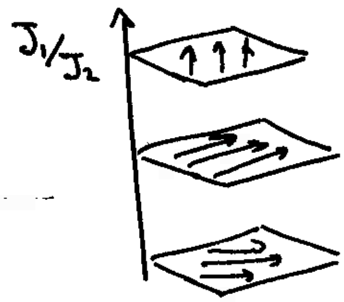
We can now turn to a more global discussion of the system of nonlinearly coupled oscillators. For definiteness, I will continue to consider the case of two degrees of freedom, for which the unperturbed Hamiltonian is

$$\mathcal{H}_0 = h_0(J_1, J_2)$$

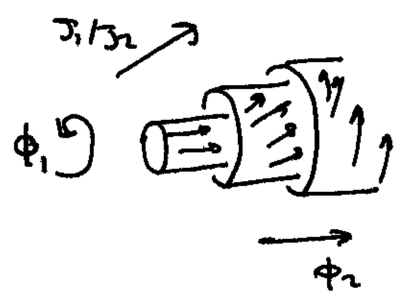
and the two angle variables evolve with frequencies

$$\omega_1 = \frac{\partial h_0}{\partial J_1} \qquad \omega_2 = \frac{\partial h_0}{\partial J_2}$$

The phase space is 4-dimensional. The surface of fixed energy is 3-dimensional. The flow generated by the unperturbed Hamiltonian consists of two-dimensional sections with fixed J_1, J_2 with constant flow in each section. These are planes covering $\phi_i \in [0, 2\pi]$ for $i = 1, 2$. Each plane is periodically connected in both directions and so is a torus. These sections are called *invariant tori.* The tori laminate the 3-dimensional space,



Another way to picture this is a stack of cylinders. Here I draw the motion so that ϕ_1 is represented as an angle.



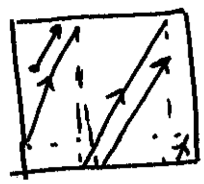
The tori are completed by connecting the figure periodically in the ϕ_2 direction.

This construction generalizes straightforwardly to systems with more than two degrees of freedom. However, in the two degree of freedom case, there is a special property. Because trajectories cannot cross in phase space, each invariant torus forms a boundary that other trajectories cannot penetrate. As we turn on a nonlinear perturbation and disrupt some of the tori, the tori that are not yet disrupted form boundaries that contain any chaotic flow. With three degrees of freedom, the tori are 3-dimensional in a constant energy surface of 5 dimensions, so it possible for chaotic trajectories to wind around the undisrupted tori.

The each torus, the unperturbed Hamiltonian gives the equations of motion

$$\dot{\phi}_1 = \omega_1 \qquad \dot{\phi}_2 = \omega_2$$

If ω_1 and ω_2 are incommensurate, a given orbit will eventually cover the whole torus.



However, there are special tori for which the ω_1/ω_2 is a rational number,

$$m\omega_1 + n\omega_2 = 0$$

Then the trajectories wind back on themselves after a finite number of traversals. Note that the ratio ω_1/ω_2 is a fixed constant on each torus.

Now turn on a nonlinear perturbation

$$V(\phi_1, \phi_2, J_1, J_2)$$

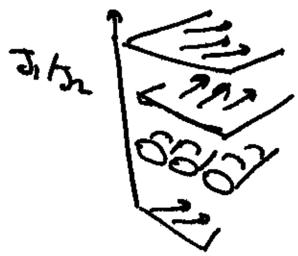
By the periodicity of ϕ_1, ϕ_2, V can be expanded in a Fourier series

$$V = \sum_{mn} e^{im\phi_1 + in\phi_2} V_{mn}(J_1, J_2)$$

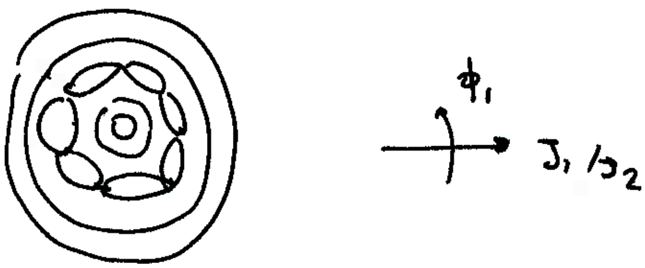
Typically, V will involve only terms with small integers m, n. The condition

$$m\omega_1 + n\omega_2 = 0$$

is the condition that the perturbation $V_{m,n}$ gives a vanishing denominator and resonant disruption of the phase space. The stack of tori with smooth translation on each layer converts to a more complex phase space flow in which a region of the stack has transverse flows similar to those of pendulum, as described above,

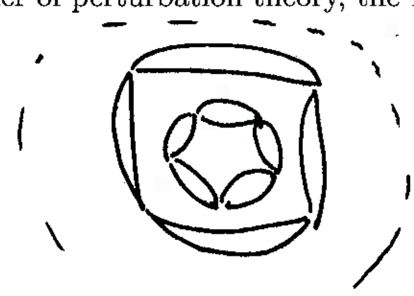


The disruption of the flow pattern is readily visible on a Poincaré section, for example, on the section $\phi_2 = 0$.



Here again, I have drawn the figure so that ϕ_1 is an angle on the plane. The original tori form circles on the Poincaré section. As we have estimated above, the first-order perturbation disrupts a region of size $\lambda^{1/2}$ around the torus for which ω_1/ω_2 satisfies the resonance condition.

At the first order of perturbation theory, the Poincaré section will have the form



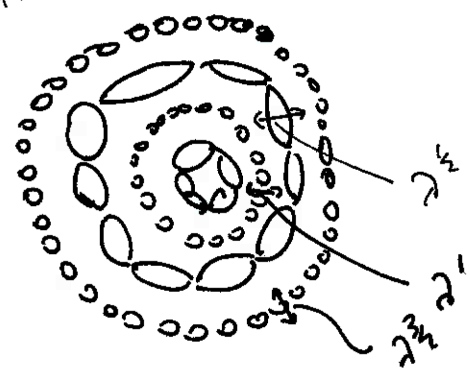
with well-separated disrupted regions bounded by invariant tori that are smooth deformations of the original tori of the unperturbed problem. The disrupted regions are associated with integers m, n such that

$$|m| + |n| < N$$

In the second order of perturbation theory, the square of the nonlinear term will generate perturbations with

$$|m| + |n| < 2N \quad .$$

These will give new resonance conditions and disrupt new tori between the previously disrupted regions. The size of the region disrupted at this stage is of order $(\lambda^2)^{1/2} = \lambda$. In third order, the perturbation will disturb additional regions, disrupting slices of phase space of size $\lambda^{3/2}$

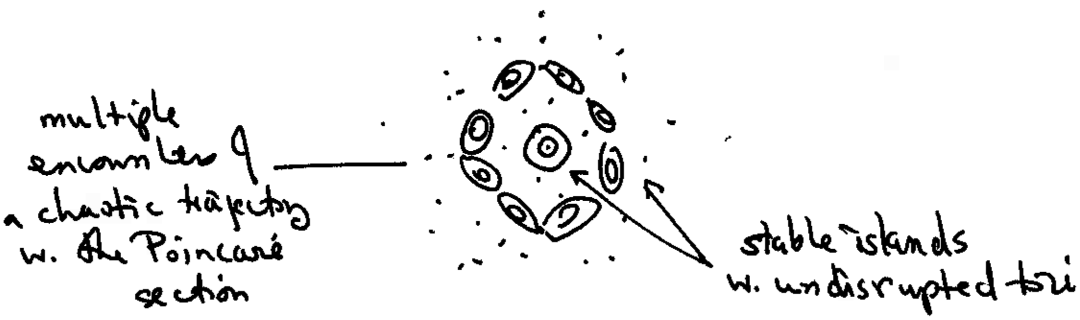


Eventually, every torus for which the ratio ω_1/ω_2 is a rational number n/m will be disrupted. However, if the required integers n and m are very large, the disruption will come only at a very high order of perturbation theory, and the associated disrupted region will be very thin. This disrupted regions are dense in the phase space. But it is still possible that, if the thickness of the regions decreases sufficiently rapidly, there is space left over in which smooth invariant tori remain.

This question was answered by Kolmogorov, Arnold, and Moser with proofs of increasing sophistication. These authors showed that, for sufficiently small λ, the sum of areas contained in the disrupted regions is *finite*. That is, for a sufficiently weak nonlinear perturbation, there are tori, necessarily with ω_1/ω_2 equal to an irrational number, that remain smooth and are not disrupted. This result is called the *KAM theorem.*

In a system with 2 degrees of freedom, the stable KAM tori form boundaries that cannot be crossed by other phase space flows. Then, the phase space motions are restricted to lie in the region between a pair of undisturbed tori.

In a system with three or more degrees of freedom, the stable KAM tori are not boundaries, and trajectories from the disrupted regions can flow around them. A typical appearance of the Poincaré section in such a system is



As the strength of the perturbation is increased, irrational tori that are closer and closer to rational numbers are disrupted. J. M. Greene proposed that the the last surviving torus would be characterized by the ratio of frequencies that is most irrational in the following sense:

Any number r can be represented uniquely as a *continued fraction* by writing

$$r = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

$$\equiv \{a_0; a_1, a_2, a_3, \ldots\}$$

with integers $a_0, a_1, a_2, \ldots$. Truncating this series after the nth term gives a rational approximation to r of the form p_n/q_n. The successive approximations obey the recursion formula

$$p_{n+1} = a_{n+1} p_n + p_{n-1}$$

$$q_{n+1} = a_{n+1} q_n + q_{n-1}$$

which implies

$$\left| r - \frac{p_n}{q_n} \right| < \frac{1}{a_{n+1} q_n^2}$$

So the successive approximations converge very rapidly, much faster than $1/q_n$, to the desired irrational number. The continued fraction representation of some familiar irrational numbers are:

$$\sqrt{2} = \{1; 2, 2, 2, \cdots\}$$

$$e = \{2; 1, 2, 1, 1, 4, 1, 1, 6, \cdots\}$$

$$\pi = \{3; 7, 15, 1, 292, 1, \cdots\}$$

For π, the successive rational approximations are

$$3, \quad \frac{22}{7}, \quad \frac{333}{106} = 3.14151 \qquad \frac{355}{113} = 3.1415929$$

Notice that, whenever a large integer appears in the continued fraction representation, truncating the series just before that value gives a very accurate approximation. By this criterion, e and π are not very irrational. The most irrational number, in the sense of being most poorly approximated by rationals, is

$$\gamma = \frac{\sqrt{5}-1}{2} = \{0; 1, 1, 1, \cdots\} = 0.61803\ldots$$

the *Golden Mean.* The first approximations are

$$\frac{1}{2}, \quad \frac{2}{3}, \quad \frac{3}{5}, \quad \frac{5}{8}, \quad \cdots$$

All of these approximations are the ratios of successive numbers in the Fibonacci series. The sequence converges to γ, but very slowly.

Thus, in a mechanical system that perturbed by a nonlinear term of increasing strength, the last KAM torus to be disrupted is that on which $\omega_1/\omega_2 = \gamma$. When this torus is destroyed, there are no natural boundaries in phase space and chaotic trajectories can roam over the whole energy surface.

Leo Kadanoff and Scott Shenker studied the process of disruption of this final KAM torus with $\omega_1/\omega_2 = \gamma$ in a system of two coupled oscillators. To approach this point, they studied periodic orbits for which ω_1/ω_2 is a ratio of Fibonacci numbers, systematically increasing the denominator. On the Poincaré section $\phi_2 = 0$, the successive intersections generate a recursion

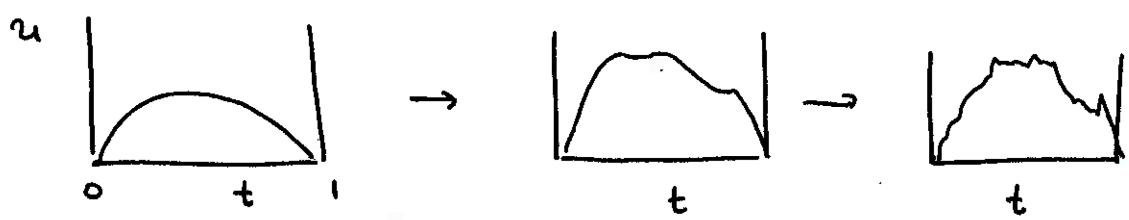
$$\phi_{1\,n+1} = \phi_{1\,n} + 2\pi u(n)$$

$$r_{n+1} = r_n + 2\pi V(n)$$

where r is the local value of ω_1/ω_2 at the point where the trajectory intersects the section. Nearby values of ϕ_n lead to nearby values of ϕ_{n+1}, so the shifts u and v are smooth functions of

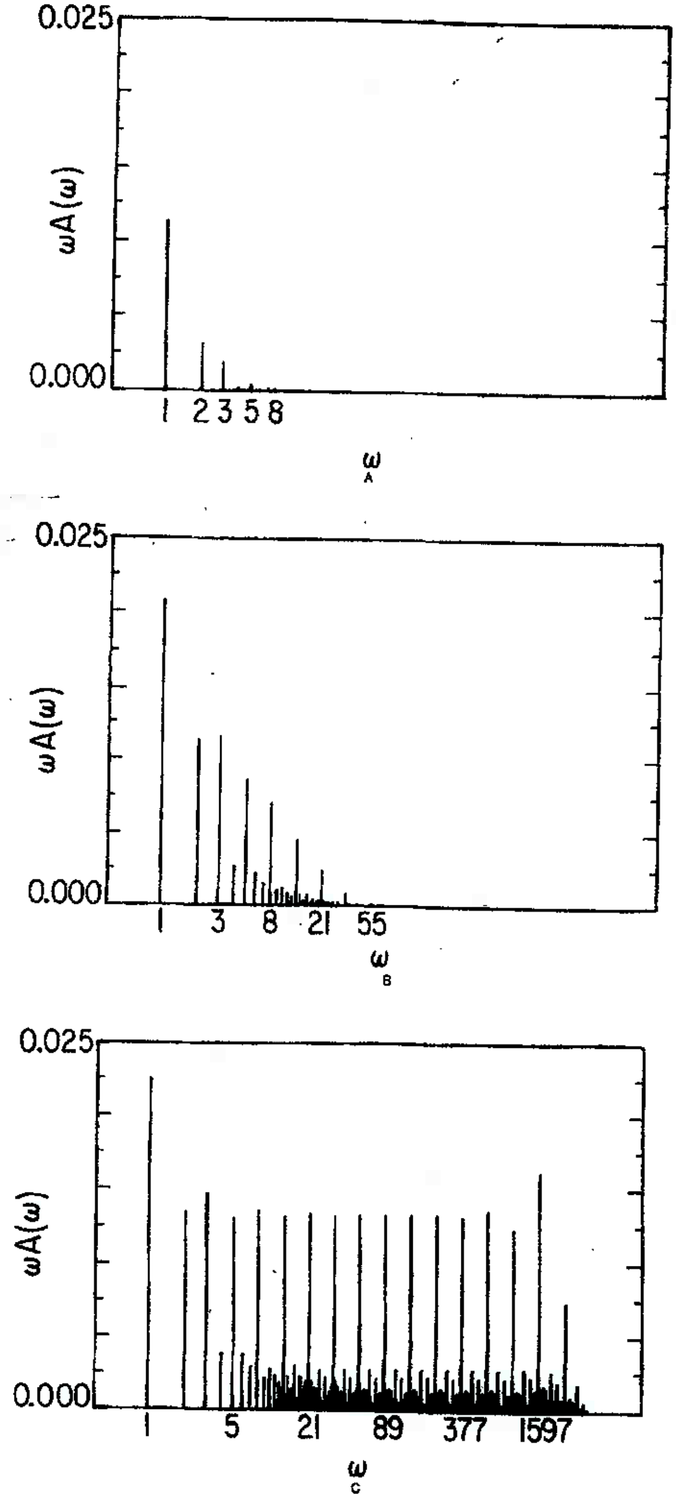
$$t = \frac{\phi_n}{2\pi} \in [0,1]$$

More properly, the functions $u(t)$, $v(t)$ are smooth for typical values of λ but develop increasingly local structure as λ approaches the critical value λ_c at which the KAM torus disappears.



This development of structure can also be seen from the Fourier components of $u(t)$, shown in the figure (taken from Shenker and Kadanoff, J. Stat. Phys. 27, 631 (1982)). The data is for a periodic trajectory with denominator $q_n = 4181$. All frequencies that appear are multiples of the fundamental frequency. However, as the transition is approached, there is structure at increasingly higher frequencies, corresponding to smaller structure in t. The high peaks are located at Fibonnaci numbers and correspond to the expected ratio of frequencies at the Golden Mean. However, there is additional structure at all allowed frequencies.

Kadanoff and Shenker investigated the behavior of the finite difference of $u(t)$ at neighboring intersection points on the periodic trajectory


0.025
0.000
ωA(ω)
1 2 3 5 8
ω
A
0.025
0.000
ωA(ω)
1 3 8 21 55
ω
B
0.025
0.000
ωA(ω)
1 5 21 89 377 1597
ω
C

$$d(t) = u(t + \frac{1}{2q_n}) - u(t)$$

For small values of λ, this difference vanishes as $q_n \to \infty$ as $1/q_n$. This implies that the function $u(t)$ is differentiable. However, at the critical value of λ,

$$d(0) \sim \frac{1}{q_n^{x_0}}$$

with exponent $x_0 = 0.721 \pm 0.001$. In fact, Kadanoff and Shenker give evidence that the behavior of the finite difference is described by a scaling function

$$d(0) \sim \frac{1}{q_n^{x_0}} f_0\left(\frac{1}{\epsilon q_n}\right)$$

where $\epsilon = \lambda_c - \lambda$ and $f_0(x)$ has the behavior

$$f_0(x) \to \begin{cases} x^{1-x_0} & x \to 0 \\ \text{const} & x \to \infty \end{cases}$$

Thus, the disappearance of the final KAM torus gives another example of the rich structure that can appear at the transition to chaos.

Much more remains to be understood about chaos, both in forming a general theory of routes to chaos and in understanding the nature of deeply chaotic motions. I encourage you to look further into this subject.