Nov. 4

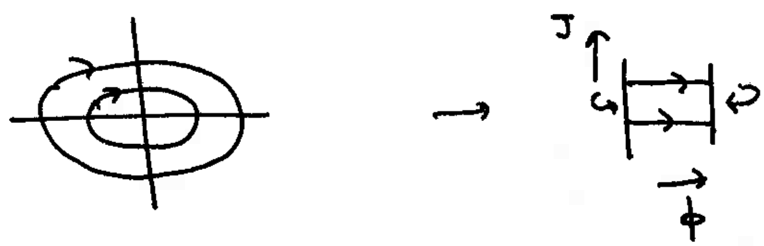
Hamilton-Jacobi Theory

In the previous lecture, I described the formal properties of canonical transformations. In this lecture, I will give some applications of canonical transformations to solve problems of mechanics.

The basic idea of these solutions can be found already in the solution of the harmonic oscillator given in the previous lecture. We found a way to canonically map the phase space variables (x, p) for the harmonic oscillator to new variables (ϕ, J) such that the Hamiltonian took the form

$$\mathcal{H} = \omega J$$

The elliptic paths of the system in phase space were transformed to straight lines on a cylinder.



It would be wonderful if such a transformation existed to simplify any mechanical system. That is, we might hope that, even for a nonlinear system with n degrees of freedom, we could always find a canonical transformation

$$(q_i, p_i) \longrightarrow (\phi_i, J_i)$$

such that the Hamiltonian took the form in the new coordinates

$$\mathcal{H} = h(J_1 \cdots J_n)$$

In particular, the Hamiltonian would not be a function of the ϕ_i. Then the equations of motion of the system would be

$$\dot{\phi}_i = \frac{\partial H}{\partial J_i} = \text{constant}$$

$$\dot{J}_i = -\frac{\partial H}{\partial \phi_i} = 0$$

with the J_i constants of the motion and the ϕ_i changing by simple translation at the speeds $\omega_i(J)$, which would be fixed functions of the constant J_i. The motions of the system would then be mapped into straight lines in a subspace with fixed values of the J_i. The new variables ϕ_i and J_i are called *angle* and *action variables.*

It is indeed too much to hope that this simplification exists for every mechanical system. However, it turns out that there do exist many systems for which there is such a simplification. These include nonlinear systems with quite nontrivial dynamics. We call such systems, with a system of n conservation laws, *integrable mechanical systems.*

In this lecture, I will carry out this program of conversion of some relatively simple mechanical systems to angle-action variables. The results will be useful in two ways, first, as a way of making the dynamics of these systems are clear as possible, second, as a starting point for the analysis of the coupling of these systems to perturbations via additional nonlinear interactions.

To begin, I consider a general 1-dimensional potential problem

$$H = \frac{p^2}{2m} + V(x)$$

As a first exercise, I will convert this Hamiltonian to new variables $\psi, \mathcal{J}$ such that

$$H = \mathcal{J}$$

This can be done with the generating function

$$F_2(x, \mathcal{J})$$

By construction, $\mathcal{J}$ will be a constant in the motion, so we should best think of F_2 simply as a function of x, with parametric dependence on the constant $\mathcal{J}$. It is conventional to write

$$F_2(x, \mathcal{J}) = W(x)$$

The function $W(x)$ is called the *Hamilton-Jacobi function.* The momentum p is given by

$$p = \frac{dW}{dx}$$

The statement that $H = \mathcal{J}$ then leads to the following differential equation for $W(x)$:

$$\mathcal{J} = \frac{1}{2m}\left(\frac{dW}{dx}\right)^2 + V(x)$$

This is called the *Hamilton-Jacobi differential equation.* In this case, the equation can be solved straightforwardly.

$$\frac{dW}{dx} = \pm\sqrt{2m(\mathcal{J} - V(x))}$$

$$W(x) = \pm\int^x dx' \left(2m\left[\mathcal{J} - V(x)\right]\right)^{\frac{1}{2}}$$

To compare this result to the analysis in the previous lecture, we can evaluate W for the case of a harmonic oscillator

$$V = \frac{1}{2} m\omega^2 x^2$$

Then (choosing the negative sign)

$$W(x) = -\int^x dx \, \left[2mJ - m^2\omega^2 x^2\right]^{1/2}$$

$$= -\frac{1}{2m\omega}\left[m\omega x \sqrt{2mJ - m^2\omega^2 x^2} - 2mJ \cos^{-1}\frac{m\omega x}{\sqrt{2mJ}} \right]$$

Our previous solution of the harmonic oscillator used the representation

$$x = \left(\frac{2J}{m\omega}\right)^{1/2} \cos\phi = \left(\frac{2J}{m\omega^2}\right)^{1/2} \cos\phi$$

This implies

$$\cos^{-1}\frac{m\omega x}{\sqrt{2mJ}} = \phi$$

Then

$$W(x) = -\frac{m\omega}{2m\omega}\left(\frac{2J}{m\omega^2}\right)^{1/2} (2mJ)^{1/2} \cos\phi \sin\phi + \frac{J}{\omega}\phi$$

$$= -\frac{m\omega x^2}{2} \tan\phi + J\phi$$

so that finally

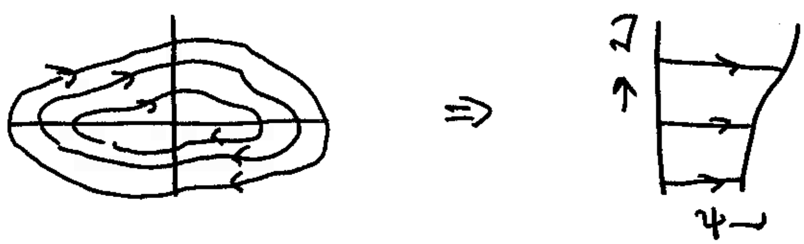
$$W(x) = F_1(x,\phi) + J\phi$$

This is the standard relation between F_1 and F_2 discussed in the previous lecture.

The change of variables that I have just described can be carried out for any potential problem. It gives the dynamical equations

$$\dot{\psi} = 1 \qquad \dot{\mathcal{J}} = 0$$

and maps the phase space to a set of straight lines traversed at constant speed,



However, it is not quite the most convenient representation, because the periodicity of the variable ψ is not obvious, and also changes with the value of the energy $\mathcal{J}$. It would be better to find a change of variables such that the coordinate variable ϕ were always periodic with period 2π. This ϕ and its conjugate momentum J are the *angle-action variables* proper.

To find these variables,, we look for a

$$W(x) = F_2(x,J)$$

that converts H to $H(J)$ such that the period the angle variable ϕ is equal to 2π for any orbit. Again, we identify the original momentum as

$$p = \frac{dW}{dx}$$

From this we derive the Hamilton-Jacobi differential equation

$$H(J) = \frac{1}{2m}\left(\frac{dW}{dx}\right)^2 + V(x)$$

and the solution

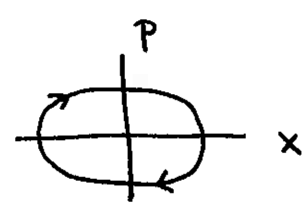
$$W(x) = \int^x dx \; \left(2m\left[H - V(x)\right]\right)^{1/2}$$

where

$$H = H(J)$$

Now we need a way to identify J. To do this, integrate $W(x)$ around a periodic orbit.

$$\Delta W = \oint dx \; p$$



Since

$$\frac{\partial W}{\partial J} = \phi$$

the change in W around a periodic orbit is

$$\frac{\partial}{\partial J} \Delta W = 2\pi$$

Then

$$\Delta W = 2\pi J$$

and we can identify

$$J = \frac{1}{2\pi} \oint p \, dx$$

More generally, the *action* variables in angle-action coordinates satisfy

$$J = \frac{1}{2\pi} \oint p \, dq$$

By solving this equation for $H(J)$, we can find the frequencies of the orbits as a function of energy. In the potential problem, the relation between H and J is given by the integral

$$J = \frac{1}{2\pi} \oint dx \; \left[2m\left(H - V(x)\right)\right]^{\frac{1}{2}}$$

We can check this easily for the case of the harmonic oscillator. Setting

$$V = \frac{1}{2} m\omega^2 x^2$$

we have

$$J = \frac{1}{2\pi} \oint dx \, \left[2mH - m^2\omega^2 x^2\right]^{1/2}$$

To evaluate the integral, set

$$x = \left(\frac{2H}{m\omega^2}\right)^{1/2} \cos\phi$$

Then

$$dx = -\left(\frac{2H}{m\omega^2}\right)^{1/2} \sin\phi \, d\phi$$

so that finally

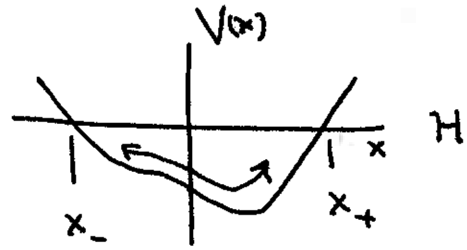
$$J = \frac{1}{2\pi} \left(\frac{2H}{m\omega^2}\right)^{1/2} \left[2mH\right]^{1/2} \int_0^{2\pi} d\phi \, \sin^2\phi = \frac{2H}{2\pi\omega} \cdot \pi$$

or

$$J = \frac{H}{\omega}$$

as before.

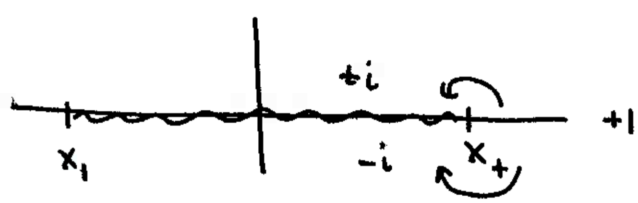
There is another, more generally applicable, way to evaluate the integral. Consider the integral over x as an integral in the complex x plane. The integrand $p(x)$ is a square root with branch points at the turning points of the motion. L:et these points be given by x_- and x_+,



If we regard the function

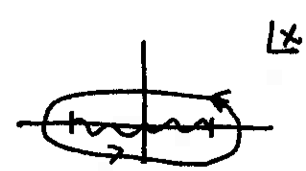
$$[V(x) - H]^{1/2}$$

as an analytic function of complex x, this function is real and positive on the real x axis for $x > x_+$ and for $x < x_-$. However, for $x_- < x < x_+$, it is imaginary. There is a branch cut from $x = x_-$ to $x = x_+$. Analytically continuing from $x > x_+$, we find that the function has phase i above the cut and phase $-i$ below the cut,



Then the Hamilton-Jacobi integral can be written

$$J = -\frac{1}{2\pi i} \oint dx \; \left(2m[V(x) - H]\right)^{1/2}$$



as an integral around a closed contour that encircles the branch cut.

If the integrand has no other singularities in the complex x plane, we can push the contour to infinity and evaluate it using the asymptotic expansion of the integrand and the relation

$$\oint \frac{dx}{x^n} = 2\pi i \cdot \begin{cases} 1 & n=1 \\ 0 & \text{otherwise} \end{cases}$$

For example, for the harmonic oscillator, the integrand is

$$[2m(V(x)-H)]^{1/2} = [(m\omega x)^2 - 2mH]^{1/2} \cong m\omega x - \frac{mH}{m\omega x} + \cdots$$

Then

$$J = -\frac{1}{2\pi i} \oint dx \left(m\omega x - \frac{mH}{m\omega x} + \cdots \right)$$

This gives

$$J = \frac{1}{2\pi i} \frac{H}{\omega} \cdot 2\pi i$$

or, once again,

$$J = \frac{H}{\omega}$$

We can repeat this exercise with the addition of a small nonlinear term to the potential

$$V(x) = \frac{1}{2} m\omega^2 x^2 + \frac{1}{4} m\lambda x^4$$

Keeping terms to first order in λ, the expansion of the integrand for large x is

$$[2m(V(x)-H)]^{\frac{1}{2}} = \left[(m\omega x)^2 - (2mH - \tfrac{1}{2}m^2\lambda x^4)\right]^{\frac{1}{2}}$$

$$= m\omega x \left[1 - \frac{1}{2}\frac{2mH - \frac{1}{2}m^2\lambda x^4}{m^2\omega^2 x^2} - \frac{1}{8}\frac{(2mH - \frac{1}{2}m^2\lambda x^4)^2}{(m^2\omega^2 x^2)^2} + \frac{1}{16}(-1)^3\frac{(2mH - \frac{1}{2}m^2\lambda x^4)^3}{(m^2\omega^2 x^2)^3} + \cdots\right]$$

or, finally,

$$= m\omega x - \frac{H}{\omega x} + \cdots + \frac{3}{16}\frac{(2mH)^2(\frac{m^2}{2}\lambda x^4)}{m^6\omega^2 x^6} + \cdots$$

Then

$$= m\omega x - \frac{H}{\omega x} + \frac{3}{8}\lambda\frac{H^2}{m\omega^5}\frac{1}{x} + \cdots$$

$$J = -\frac{1}{2\pi i}\oint dx\left[\cdots - \frac{H}{\omega x} + \frac{3}{8}\lambda\frac{H^2}{m\omega^5}\frac{1}{x} + O(\lambda^2) + \cdots\right]$$

$$= \frac{H}{\omega} - \frac{3}{8}\frac{\lambda H^2}{m\omega^5} + \cdots$$

Solving this equation for H,

$$H = \omega J + \frac{3}{8}\lambda\frac{J^2}{m\omega^2}$$

Then the frequency of the anharmonic oscillator is given by

$$\bar{\omega}(J) = \frac{dH}{dJ} = \omega + \frac{3}{4}\lambda \frac{J}{m\omega^2}$$

or, finally,

$$A = \left(\frac{2J}{m\omega}\right)^{1/2}$$

$$\bar{\omega}(J) = \omega + \frac{3}{8}\frac{\lambda}{\omega} A^2 + \cdots$$

This is the same result that we found in an earlier lecture using an *ad hoc* rearrangement of a secular perturbation.

Hamilton-Jacobi theory generalizes to multi-dimensional problems that are integrable, that is, problems that have enough conservation laws to reduce the dynamics to orbit equations. Such problems can often be solved by *separation of variables*, that is, by writing W in the form

$$W(q_1, q_2 \ldots) = W_1(q_1) + W_2(q_2) + \cdots$$

where each term depends parametrically on the full of set of J_i, which, however, are all constants.

A simple example is given by the problem of particle motion under a central force in 2 dimensions. This problem has two conservations laws of energy and angular momentum, in all 2 conservation laws for 2 degrees of freedom. The Hamiltonian is

$$H = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} + V(r)$$

The polar coordinate ϕ is already an angle variable, and the conjugate momentum p_ϕ is conserved, so it can be considered as an action variable. Now write

$$W(r,\phi) = W_r(r) + W_\phi(\phi)$$

The ϕ coordinate is treated by making $W_\phi(\phi)$ the identity transformation

$$W_\phi = p_\phi \cdot \phi$$

Then, around an orbit in ϕ,

$$\Delta W = 2\pi p_\phi$$

as required.

The Hamilton-Jacobi differential equation for this system is

$$H = \frac{1}{2m}\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial W}{\partial \phi}\right)^2 + V(r)$$

Introducing the expression for W with separation of variables, and setting $W_\phi = p_\phi \phi$, we find

$$H = \frac{1}{2m}\left(\frac{dW_r}{dr}\right)^2 + \frac{p_\phi^2}{2mr^2} + V(r)$$

Then

$$W_r = \int^r dr \left[2m\left(H - \frac{p_\phi^2}{2mr^2} - V(r)\right)\right]^{\frac{1}{2}}$$

This generating function converts the r dynamics to angle-action variables. In all, we have the change of variables

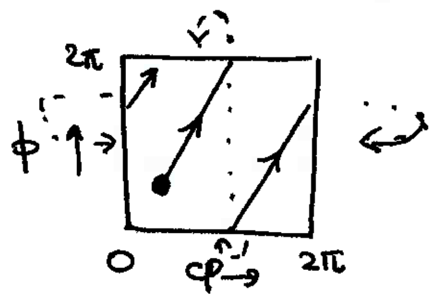
$$(r\ \phi\ p_r\ p_\phi) \to (\varphi, \phi, J, J_\phi = p_\phi)$$

with the action variable for the r motion satisfying

$$J = \frac{1}{2\pi}\oint dr\ \left[2mH - \frac{p_\phi^2}{r^2} - 2mV(r)\right]^{1/2}$$

The new Hamiltonian is a function of the conserved momenta J, J_ϕ only.

We have now mapped the phase space of the problem into a torus



or, rather, a tower of tori, one for each fixed value of (J, J_ϕ). The orbit that I have drawn on the torus is the image of an orbit that looks like

in the real 2-dimensional coordinate space.

Canonical transformations have the power to reduce problems with complex patterns of precession and nutation to triviality. It is interesting to ask, though, whether there are motions whose intrinsic complexity eventually defeats the power of these transformations. As we continue in the course, we will discuss systems with a transition to chaos, where in end the true complexity of the motion wins out.