Oct. 12

The Rotation Group

The next major topic in the course is the study of general rotational motions of bodies in 3 dimensions. To prepare for this subject, it will be very useful to discuss in detail how one parametrizes 3-dimensional rotations. This subject has many applications within classical mechanics, from the motion of bacteria to the motion of galaxies, and also provides the foundation for the discussion of angular momentum in quantum mechanics.

Rotations form a *group*, and, more precisely, a *continuously generated compact group.* I will begin by defining all of these terms and explaining how they apply to rotations.

A *group* is a set elements a with a binary operation $\circ$ satisfying the properties:

1. Associativity: $a \circ (b \circ c) = (a \circ b) \circ c$.
2. Existence of an identity: There exists $\mathbf{1}$ such that, for all a, $\mathbf{1} \circ a = a \circ \mathbf{1} = a$.
3. Existence of an inverse: For any a, there exists an element b such $a \circ b = b \circ a = \mathbf{1}$.

A continuously generated group, also called a *Lie group*, is a group in which the operation of a group element can be built up by from infinitesimal operations, that is, operations that are close to the identity. We can represent an infinitesimal group element formally by writing

$$g = \mathbf{1} + i\alpha \mathbf{T} + \mathcal{O}(\alpha^2)$$

where $\mathbf{1}$ is the identity. (The factor i is for later convenience.) Applying this operation a large number of times, we build up the finite operation

$$\lim_{N\to\infty} \left(\mathbf{1} + \frac{i\alpha}{N}\mathbf{T}\right)^N = e^{i\alpha \mathbf{T}} = \mathbf{1} + i\alpha \mathbf{T} + \frac{1}{2}(i\alpha)^2 \mathbf{T}^2 + \dots$$

The exponential series converges for any value of a, however large. Thus, we can work the these exponentials of aT as formal power series in T. The element T is called a *generator* of the group. From the set of generators, we can choose a maximal set of linearly independent generators

$$\{T_i\}$$

If the number of these elements is finite, we have a *finite-dimensional Lie group.*

We can now represent a finite group element as

$$g = 1 + a_i T_i + a_{ij} T_i T_j + \cdots$$

We can definite the multiplication law for general group elements if we know the rule for putting any string of T_i's, for example,

$$T_1 T_2 T_2 T_1 T_3 \cdots$$

into a canonical order, for example, as linear combinations of products

$$(T_1)^2 \, T_2 \, (T_3)^3 \cdots$$

To reorder generators T_i, we only need to know their commutation relations $[T_i, T_j]$. The commutator of two generators must be another generator in the set, so we can specify the commutator by writing

$$[T_i, T_j] = T_i T_j - T_j T_i = i f^{ijk} T_k$$

We can work out the multiplication law for any finite group elements $G_1 \circ G_2$ from the values of the f_{ijk}. The equation above is called the *Lie algebra* associated with the Lie group.

The f_{ijk} are called the *structure constants* of the Lie group. From the definition, f_{ijk} is antisymmetric in i, j. It can be shown that, if the group action spans a *finite dimensional compact manifold*, then it is possible to choose normalizations for the T_i so that f_{ijk} is totally antisymmetric in its three arguments.

The most familiar examples of Lie groups are groups of matrices acting on one another by matrix multiplication. The simplest algebra of matrices is that of multiplication of general $n \times n$ matrices. This algebra is called $GL(n)$ (*general linear* transformations). Matrix multiplication is associative, and it has an identity, the $n \times n$ matrix $\mathbf{1}$. A general matrix does not have an inverse, but matrices sufficiently close to the identity are invertible, since

$$g = (1 + \alpha)_{ij}$$

has inverse

$$g^{-1} = (1 - \alpha)_{ij}$$

to order α. As a manifold, $GL(n)$ is a noncompact, and, indeed, hugely unbounded.

An $n \times n$ matrix is a linear transformation of an n component vector

$$\xi'_\alpha = M_{\alpha\beta} \xi_\beta$$

To define a group of matrices, and to make the group space compact, we can restrict the set of matrices that we consider to those that preserve some property of the vector ξ. An immediate choice is to consider complex-valued vectors and choose matrices that preserve the inner product

$$\eta^\dagger \xi = \sum_\alpha \eta^*_\alpha \xi_\alpha$$

These operations do form a group: If M preserves the inner product, it will preserve the length of any vector ζ. If ζ is an eigenvector of M,

$$\zeta' = M\zeta = \lambda\zeta$$

then

$$\zeta'^\dagger \zeta' = \zeta^\dagger M^\dagger M \zeta = \lambda^* \zeta^\dagger \lambda \zeta = |\lambda|^2 \zeta^\dagger \zeta$$

If this expression is equal to $\zeta^\dagger\zeta$, then

$$|\lambda|^2 = 1 \quad \text{or} \quad \lambda = e^{i\beta} \quad \text{for some } \underline{\text{real}}\ \beta$$

Then M has no zero eigenvectors and thus is invertible. In addition, the space of vectors with given inner product is compact, so this subset of $GL(n)$ forms a compact group. This group is called the group of $n \times n$ *unitary* transformations, or $U(n)$. A unitary matrix can be characterized by considering the transformation of the innner product for general vectors η and ξ. The transformation of the inner product is

$$\eta^\dagger \xi \to \eta^\dagger M^\dagger M \xi$$

so if the product is to be unchanged for any vectors

$$M^\dagger M = 1$$

or

$$M^\dagger = M^{-1}$$

for any unitary matrix.

$U(n)$ contains a trivial subgroup which multiplies any ξ by a phase

$$\xi \rightarrow e^{i\gamma}\xi$$

It is conventional to extract this subgroup by placing the further restriction on M

$$\det M = 1$$

The set of $n \times n$ matrices M that preserve the inner product $\eta^\dagger\xi$ and also have determinant 1 comprise the *special unitary* group $SU(n)$.

The infinitesimal elements of $SU(n)$ also can be characterized simply. Consider the transformation of the inner product under an infinitesimal matrix transformation.

$$\eta^\dagger\xi \rightarrow \eta^\dagger(1 - i\alpha_i T_i^\dagger)(1 + i\alpha_j T_j)\xi$$

$$= \eta^\dagger(1 + i\alpha_i(T_i - T_i^\dagger) + \cdots)\xi$$

The equals $\eta^\dagger\xi$ for general η, ξ only if T_i is Hermitian

$$T_i^\dagger = T_i$$

There are n^2 independent $n \times n$ Hermitian matrices. Of these, one is

$$T_0 = \mathbb{1}$$

which generates the phase multiplication

$$\xi \to e^{i\gamma}\xi$$

The matrices orthogonal to this one satisfy

$$\mathrm{tr}\, T_i = 0$$

Matrices satisfying this condition generate $SU(n)$ elements: Since

$$\begin{aligned} \det(1 + i\alpha_i T_i) &= \prod_a \lambda_a = \exp\left(\sum_a \log \lambda_a\right) \\ &= \exp\left[\mathrm{tr} \log(1 + i\alpha_i T_i)\right] \\ &= 1 + i\alpha_i \,\mathrm{tr}\, T_i \qquad + O(\alpha) \end{aligned}$$

a transformation generated by such a T_i has determinant 1.

By placing stronger conditions on the matrices M of $SU(n)$, it is possible to define additional matrix groups as subgroups of $SU(n)$. There are two natural subgroups obtained in this way. First, consider elements of $SU(n)$ which also preserve the inner product of η and ξ without complex conjugation

$$\eta \cdot \xi = \sum_\alpha \eta_\alpha \xi_\alpha$$

This is the usual dot product of vectors, and so the matrices of this class will be n-dimensional rotations. In finite form, the action of M on this inner product is

$$\eta \cdot \xi \rightarrow \eta M^T M \xi$$

so invariance of the inner product requires

$$M^T M = 1 \quad \text{or} \quad M^T = M^{-1}$$

In infinitesimal form, the action is

$$\eta \cdot \xi \rightarrow \eta (1 + i\alpha_i (T_i)^T)(1 + i\alpha_i T_i) \xi$$

$$= \eta (1 + i\alpha_i (T_i + T_i^T)) \xi$$

Thus, a T_i generates this subgroup if it is both Hermitian and *antisymmetric.* With both restrictions, T_i must have the form

$$T_i = -i R_i$$

where R_i is a *real, antisymmetric* $n \times n$ matrix. A real antisymmetric matrix has

$$\frac{n(n-1)}{2}$$

independent components. We thus find a group with $n(n-1)/2$ generators. This group is called $SO(n)$, the *special orthogonal* group.

A typical generator of $SO(n)$ has the form

$$\mathcal{R} = \begin{pmatrix} 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 0 & \\ -1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \\ & & & & 0 \end{pmatrix}$$

The corresponding infinisimal action on a vector ξ is

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \vdots \end{pmatrix} \to (1 + \alpha \mathcal{R}) \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \vdots \end{pmatrix}$$

In the case shown,

$$\begin{aligned} \xi_1 &\to \xi_1 + \alpha \xi_3 \\ \xi_2 &\to \xi_2 \\ \xi_3 &\to \xi_3 - \alpha \xi_1 \\ &\vdots \end{aligned}$$

This is a rotation in the (1,3) plane. The generator with nonzero elements in the (k,ℓ) and (ℓ,k) positions similarly generates a rotation in the (k,ℓ) plane. The number of orthogonal planes through the origin in n dimensions is $n(n-1)/2$, so all of the generators of $SO(n)$ can be interpreted in this way.

A generalization of the condition that M preserves the vector dot product is a condition that M preserves a positive symmetric quadratic form

$$\eta_\alpha A_{\alpha\beta} \xi_\beta$$

However, this does not produce a distinct new subgroup. Any such A can be diagonized

$$A = V^T \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & \ddots \end{pmatrix} V = W^T W$$

with

$$W = \begin{pmatrix} \sqrt{a_1} & & \\ & \sqrt{a_2} & \\ & & \ddots \end{pmatrix} V$$

Then the new subgroup is just $SO(n)$ acting on the vectors $W\xi$.

A different generalization that does lead to a new structure is to take the quadratic form A to be *antisymmetric*. For clarity, we should first change the basis so that the A takes its simplest form. Assuming that n is *even*, the most useful form is

$$\eta_\alpha E_{\alpha\beta} \xi_\beta \qquad \text{with} \qquad E = \left(\begin{array}{c|c} 0 & \mathbb{1} \\ \hline -\mathbb{1} & 0 \end{array}\right)$$

$$n/2 \times n/2 \text{ blocks}$$

Note that

$$E^2 = -1 \qquad E^T = E$$

We will see this structure arise in mechanics later in the course.

The subgroup of $SU(n)$ that leaves this antisymmetric inner product invariant is called the *symplectic group* $Sp(n)$. The group $Sp(n)$ exists for n even only. Its finite action on the inner product is

$$\eta^T E \xi \rightarrow \eta^T M^T E M \xi$$

so the inner product is invariant if

$$M^T E M = E \quad \Rightarrow \quad E M^T E M = -1$$

that is,

$$M^{-1} = - E M^T E$$

The action of an infinitesimal generator on the inner product is

$$\eta^T E \xi \quad \to \quad \eta^T (1 + i\alpha_i T_i^T) E (1 + i\alpha_j T_j) \xi$$

$$= \eta^T (E + i\alpha_i (E T_i + T_i^T E)) \xi$$

$$= \eta^T (E + i\alpha_i [(E T_i) - (E T_i)^T]) \xi$$

so the inner product is invariant only if (ET_i) is a *symmetric* Hermitian matrix. The group $Sp(n)$ then has $n(n+1)/2$ independent generators. Many properties of $Sp(n)$ can be found by replacing $n \to -n$ in the corresponding formulae for $SO(n)$.

We have now found three countably infinite families of compact finite-dimensional Lie groups—$SU(n)$, $SO(n)$, and $Sp(n)$. These are called the *classical Lie group.* Are there more Lie algebras not derived from these? This question was answered at the end of the 19th century of Wilhelm Killing and Élie Cartan. Their analysis made use of the *Jacobi identity,* a property of any Lie algebra. For any matrices T_i, T_j, T_k, it is automatic that

$$0 = [T_i [T_j, T_k]] + [T_j [T_k, T_i]] + [T_k [T_i, T_j]]$$

The proof is easy: Take a large piece of paper, write the right-hand side at the top, expand the commutators, and cancel terms. This equation implies an identity for the structure constants f_{ijk},

$$0 = f^{ilm} f^{jkl} + f^{jlm} f^{kil} + f^{klm} f^{ijl}$$

That constraints , quadratic in the f_{ijk}, puts constraints on the values of these quantities. The structure constants of the classical Lie algebras solve these constraints. It turns out that there are a small number of additional solutions—exactly 5 in all. These are the *exceptional Lie algebras*, called G_2, F_4, E_6, E_7, E_8. These groups are quite exotic, but several of them, in particular G_2, E_6, and E_8 play important roles in particle physics and string theory.

The most important case of $SO(n)$ is the rotation group in 3 dimensions, $SO(3)$ The generators are

$$T_1 = -i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad T_2 = -i \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad T_3 = -i \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The Lie algebra is

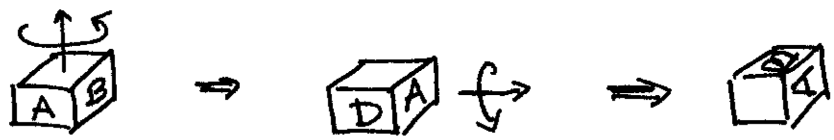
$$[T^1, T^2] = i T^3$$
$$[T^2, T^3] = i T^1$$
$$[T^3, T^1] = i T^2$$

that is

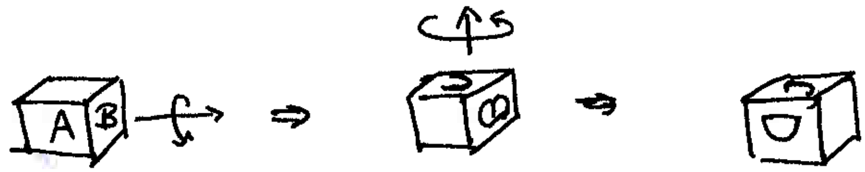
$$[T_i, T_j] = i \epsilon^{ijk} T_k$$

Note that this representation already gives a basis in which the structure constants are totally antisymmetric. You might recognize this equation as the commutation relation of angular momentum in quantum mechanics. That is not an accident, since in quantum mechanics the components of angular momentum are exactly the generators of 3-dimensional rotations.

Now that we understand the abstract structure of $SO(3)$, our next goal will be to find useful coordinate systems for describing rotational motion. This is not so straightforward. The group of 3-dimensional rotations is not communtative, so the order in which rotations are applied matters. For example, a rotation by 90° about $\hat{z}$ followed by a rotation of 90° about $\hat{y}$



and a rotation by 90° about $\hat{y}$ followed by a rotation of 90° about $\hat{z}$



put an object into completely different orientations. Thus, when we describe rotations, we must be meticulous about the order of operations. Any parametrization of rotations must take account of this. I will now describe three such parametrizations: dyadic, Euler, and Cayley-Klein. From here on, I will write the 3×3 real rotation matrix as R_{ij}.

The simplest way to parametrize a rotation is to choose an axis and represent the rotation by giving this axis together with the angle of rotation. Euler proved that this description is possible for any element of $SO(3)$: As we saw above, any $SU(3)$ matrix M has eigenvalues of the form

$$e^{i\gamma}$$

which, since $\det M = 1$, satisfy

$$e^{i\gamma_1}\, e^{i\gamma_2}\, e^{i\gamma_3}$$

The $SO(3)$ subgroup of $SU(3)$ consists of *real-valued* matrices. Then, if ξ is an eigenvector with eigenvalue $e^{i\gamma}$, then ξ^* is also an eigenvector with eigenvalue $e^{-i\gamma}$,

$$M\xi = e^{i\gamma}\xi \quad \Rightarrow \quad M\xi^* = e^{-i\gamma}\xi^*$$

Thus, an SO(3) matrix must have its three eigenvalues of the form

$$e^{i\theta}, \quad e^{-i\theta}, \quad 1$$

for some θ. The eigenvector with eigenvalue 1 is the axis of rotation; vectors parallel to this axis are not transformed. If we alight our coordinate system so that the $\hat{z}$ axis points along this vector, M takes the form

$$M = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ +\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For a more general orientation $\hat{n}$ of the axis of rotation, we can write the corresponding matrix as

$$M_{\alpha\beta}x_\beta = \Big[\cos\theta\,\vec{x} + \sin\theta\,\hat{n}\times\vec{x} + (1-\cos\theta)\,\hat{n}\,\hat{n}\cdot\vec{x}\Big]_\alpha$$

or

$$M_{ik} = \delta_{ik}\cos\theta + \epsilon_{ijk}\hat{n}^j\sin\theta + \hat{n}_i\hat{n}_k(1-\cos\theta)$$

Notice that the reflection

$$\hat{n} \longrightarrow -\hat{n}$$

is equivalent to a change in θ,

$$\theta \rightarrow -\theta \qquad \simeq \qquad \theta \rightarrow 2\pi - \theta$$

Thus, all rotations are described by this formula if we allow an arbitrary choice of $\hat{n}$ and an arbitrary θ in the region

$$0 \leq \theta < \pi$$

For a rotation about the $\hat{z}$ axis, the infinitesimal form of the rotation is

$$\overleftrightarrow{R}_z(\phi)\,\vec{x} \cong \vec{x} + \delta\phi \;\; \hat{z} \times \vec{x}$$

For a more general rotation axis, this equation reads

$$\overleftrightarrow{R}\,\vec{x} = \vec{x} + \delta\theta \;\; \hat{n} \times \vec{x}$$

For a particle in arbitrary rotational motion, we can describe the change in the particle's position in a small time interval Δt as the application to the particle's position vector of an additional infinitesimal rotation. Since any infinitesimal rotation has a definite axis and angle, we can write the particle's instantaneous velocity as

$$\vec{V} = \frac{\Delta\theta}{\Delta t}\, \hat{n} \times \vec{X}$$

It is convenient to represent this in terms of an *angular velocity vector* $\vec{\omega}$.

$$\vec{V} = \vec{\omega} \times \vec{X} \qquad \text{or} \qquad V^i = \epsilon^{ijk}\, \omega^j X^k$$

For a time-dependent rotation about a fixed axis, $\vec{\omega}$ will be oriented along this axis. For a time-dependent rotation about $\hat{z}$,

$$\frac{d}{dt} R_z(\phi) = \frac{d}{dt} \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \dot{\phi} \begin{pmatrix} -\sin\phi & -\cos\phi & 0 \\ \cos\phi & -\sin\phi & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= -\dot{\phi} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} = -i\dot{\phi}\, T_3\, R_z(\phi)$$

where T_3 is the generator of rotations about $\hat{z}$. Using the explicit form of T_3, this can also be written

$$\frac{d}{dt} \left(R_z(\phi)\right)_{ik} = \left(\epsilon^{i3j}\, \dot{\phi}\right) R_{jk}(\phi)$$

For a general rotational motion, this equation will generalize to

$$\frac{d}{dt} R_{ik} = \left(\epsilon^{ijl}\, \dot{\theta}\, \hat{n}^j\right) R_{lk}$$

Then if

$$\vec{x}(t) = \overleftrightarrow{R}(t)\,\vec{x}_0$$

the velocity of the motion is given by

$$\frac{d}{dt}\vec{x}(t) = \dot{\theta}\,\hat{n} \times \vec{x}(t)$$

in accord with the equation above.

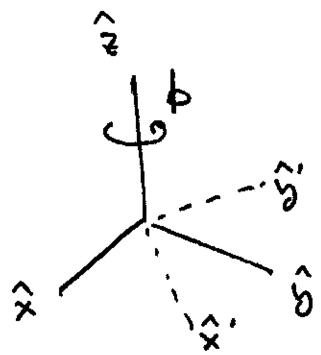
Euler gave an alternative parametrization of a rotation that often leads to a simpler description. Euler suggested that an arbitrary 3-dimensional rotation could be built up from three relatively simple rotations, in the following way: We can think of an arbitrary rotation as a reorientation of the basic coordinate axes

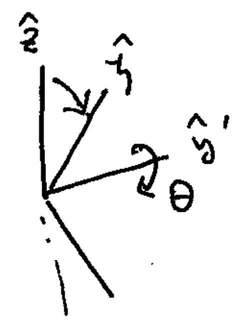
so that

$$\hat{\xi} = R\,\hat{x} \qquad \hat{\eta} = R\,\hat{y} \qquad \hat{\zeta} = R\,\hat{z}$$

Since R preserves the vector product, $\hat{\xi}$, $\hat{\eta}$, and $\hat{\zeta}$ are orthonormal unit vectors. Since R is continuously generated from the right-handed coodinate system $\hat{x}$, $\hat{y}$, $\hat{z}$, these also form a right-handed coordinate system. We can then specify the rotation R by designing a matrix that takes $\hat{x}$, $\hat{y}$, $\hat{z}$ into $\hat{\xi}$, $\hat{\eta}$, $\hat{\zeta}$.

First, arrange that $\hat{\zeta}$ is put into the correct orientation. To do this, rotate by ϕ about $\hat{z}$ and then by θ about the *new* $\hat{y}$ axis,

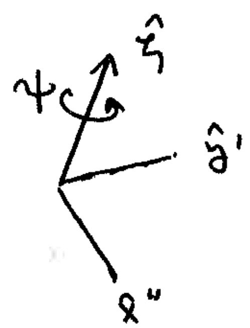




This set of operations brings $\hat{z} = (0, 0, 1)$ into the position

$$\hat{\eta} = (\cos\phi \sin\theta, \ \sin\phi \sin\theta, \ \cos\theta)$$

Now all that remains is to bring the images of $\hat{x}$ and $\hat{y}$ into their correct positions orthogonal to $\hat{\zeta}$. We can accomplish this by rotating about $\hat{\zeta}$ by an angle ψ,



In all

$$R = (\text{rotation by } \psi \text{ about } \hat{\eta}) \cdot (\text{rotation by } \theta \text{ about } \hat{y}')$$
$$\cdot\, (\text{rotation by } \phi \text{ about } \hat{z})$$

The vector $\hat{y}'$, which gives the intersection of the $\hat{x}$, $\hat{y}$ plane with the $\hat{\xi}$, $\hat{\eta}$ plane, is called the *line of nodes.*

From this description, we can obtain an explicit matrix representation of R. A rotation about $\hat{z}$ is written

$$R_z(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and, similarly, a rotation about $\hat{y}$ is written

$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

To describe a rotation about the line of nodes $\hat{y}'$, we can first rotate about $\hat{z}$ to return $\hat{y}'$ to $\hat{y}$, then rotate about $\hat{y}$, then reverse the first transformation to take $\hat{y}$ back to $\hat{y}'$. Explicitly,

$$R_{y'}(\theta) = R_z(\phi)\, R_y(\theta)\, R_z^{-1}(\phi)$$

In a similar way, we can write an explicit formula for the rotation about $\hat{\zeta}$ as

$$R_\eta(\psi) = R_{y'}(\theta) R_z(\phi)\, R_z(\psi)\, [R_{y'}(\theta) R_z(\phi)]^{-1}$$

This is all simplified with a little algebra

$$\begin{aligned} R(\psi,\theta,\phi) &= R_\eta(\psi)\, R_{y'}(\theta)\, R_z(\phi) \\ &= [R_{y'}(\theta)\, R_z(\phi) R_z(\psi)\, R_z^{-1}(\phi)\, R_{y'}^{-1}(\theta)] R_{y'}(\theta) R_z(\phi) \\ &= R_{y'}(\theta)\, R_z(\phi)\, R_z(\psi) \\ &= R_z(\phi)\, R_y(\theta)\, R_z^{-1}(\phi)\, R_z(\phi) R_z(\psi) \end{aligned}$$

The final result is

$$R(\psi,\theta,\phi) = R_z(\phi)\, R_y(\theta)\, R_z(\psi)$$

That is, the Euler parametrization of rotations is exactly the produce of rotations by ϕ, θ, and ζ about the *original* axes $\hat{z}$ and $\hat{y}$, written in just the *opposite* of the order one might have expected.

The whole space of rotations is covered if the three Euler angles are taken over the regions $0 \leq \phi \leq 2\pi$, $0 \leq \theta \leq \pi$, $0 \leq \psi \leq 2\pi$.

We can differentiate this formula to find a representation for the angular velocity in terms of time derivatives of the Euler angles. The time derivative is

$$\frac{d}{dt}[R(\psi,\theta,\phi)]_{i\ell} = \frac{d}{dt}\left[R_z(\phi)R_y(\theta)R_z(\psi)\right]_{i\ell}$$

$$= \epsilon^{ijk}(\dot{\phi}\hat{z})_j\,[R_z(\phi)R_y(\theta)R_z(\psi)]_{k\ell} + (R_z(\phi))_{ik}\,\epsilon^{kjm}(\dot{\theta}\hat{y})_j\,[R_y(\theta)R_z(\psi)]_{m\ell}$$

$$+ [R_z(\phi)\,R_y(\theta)]_{ik}\,\epsilon^{kjm}(\dot{\psi}\hat{z})_j\,[R_z(\psi)]_{m\ell}$$

The first term gives a contribution to $\vec{\omega}$ directed along the $\hat{z}$ axis. To interpret the second and third terms, we need to move the infinitesimal elements to the left so that they act on the entire rotation matrix. The rotation by ϕ about $\hat{z}$ converts $\hat{y}$ to $\hat{y}'$, so

$$(R_z(\phi))_{ik}\;\epsilon^{kjm}(\dot{\theta}\hat{y})_j\,(R_z^{-1}(\phi))_{m\ell}$$

$$= \epsilon^{ij\ell}\;(\dot{\theta}\hat{y}')_j$$

The rotation

$$R_z(\phi)\,R_y(\theta)$$

converts $\hat{z}$ to $\hat{\zeta}$,

$$[R_z(\phi)\, R_y(\theta)]_{ik}\ \epsilon^{kjm} (\dot{\psi}\hat{z})_j\ [R_{y}^{-1}(\theta)\, R_z^{-1}(\phi)]_{ml}$$

$$= [R_{y'}(\theta)\, R_z(\phi)]_{ik}\ \epsilon^{kjm} (\dot{\psi}\hat{z})_j\ [R_z^{-1}(\phi) R_{y'}^{-1}(\theta)]_{ml}$$

$$= \epsilon^{ijl} (\dot{\psi}\hat{\eta})_j$$

Thus, the time derivative of the rotation can be rewritten as

$$\frac{d}{dt}[R(\psi,\theta,\phi)]_{il} = \epsilon^{ijk}[\dot{\phi}\hat{z} + \dot{\theta}\hat{y}' + \dot{\psi}\hat{\eta}]_j\, R_{kl}(\psi,\theta,\phi)$$

If $\vec{x}(t)$ is executing rotational motion,

$$\vec{x}(t) = \overleftrightarrow{R}(\psi,\theta,\phi)\cdot\vec{x}_0$$

then

$$\frac{d}{dt}\vec{x}(t) = [\dot{\phi}\hat{z} + \dot{\theta}\hat{y}' + \dot{\psi}\hat{\eta}] \times \vec{x}(t)$$

The axes $\hat{y}'$ and $\hat{\zeta}$ are related to axes fixed in space by the relations

$$\hat{y}' = \cos\phi\, \hat{y} - \sin\phi\, \hat{x}$$

$$\hat{\eta} = \cos\phi \sin\theta\, \hat{x} + \sin\phi\sin\theta\, \hat{y} + \cos\theta\, \hat{z}$$

So, with respect to axes fixed in space, $\vec{\omega}$ takes the form

$$\vec{\omega} = \hat{x}\,[\dot{\psi}\cos\phi\sin\theta - \dot{\theta}\sin\phi]$$
$$+ \hat{y}\,[\dot{\psi}\sin\phi\sin\theta + \dot{\theta}\cos\phi]$$
$$+ \hat{z}\,[\dot{\psi}\cos\theta + \dot{\phi}]$$

In a similar way, we can obtain an expression for $\vec{\omega}$ in terms of the new axes $\hat{\xi}$, $\hat{\eta}$, $\hat{\zeta}$. We need the formulae

$$\hat{z} = \cos\theta\,\hat{\zeta} - \sin\theta\cos\psi\,\hat{\xi} + \sin\theta\sin\psi\,\hat{\eta}$$

$$\hat{y}' = \hat{\eta}\cos\psi + \hat{\xi}\sin\psi$$

Using these relations,

$$\vec{\omega} = \hat{\xi}\,[-\dot{\phi}\sin\theta\cos\psi + \dot{\theta}\sin\psi]$$

$$+ \hat{\eta}\,[\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi]$$

$$+ \hat{\zeta}\,[\dot{\phi}\cos\theta + \dot{\psi}]$$

If $\theta = 0$, the rotations by the angles ϕ and ψ have the same effect. You can see that, in the limit $\theta \to 0$, the formulae for $\vec{\omega}$ depend on the combination $(\dot{\phi} + \dot{\psi})$ only.

There is one more useful parametrization of the rotation group, due to Cayley and Klein. This parametrization begins from the representation

$$R_n(\theta) = \exp[-i\,\theta(\hat{n})_i T_i]$$

in which the T_i are matrices that obey

$$[T_i, T_j] = i\epsilon_{ijk} T_k$$

In the representations above, we used the canonical 3×3 representation of the T_i. However, the multiplication law of the rotation matrices R is specified once the commutation relations of the T_i are specified, so in principle we could use any other set of Hermitian matrices that satisfies the same commutation relations.

The Pauli sigma matrices are the three Hermitian matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These obey

$$[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k$$

The complete multiplication law of the $\vec{\sigma}$ matrices is

$$\sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k$$

Thus, the matrices

$$\sigma^i/2$$

obey the commutation relations of the generators of the rotation group. Then we can use these matrices to build a representation of the rotation matrices,

$$U_{\hat{n}}(\vec{\theta}) = \exp\left[-i\theta \frac{\vec{\sigma}}{2}\cdot\hat{n}\right]$$

To see how this works, take a 3-vector and dot it with the sigma matrices to form a 2×2 traceless Hermitian matrix

$$\vec{W}\cdot\vec{\sigma} = \underset{\sim}{W} \qquad\qquad W^i = \tfrac{1}{2}\,\mathrm{tr}\,\sigma^i \underset{\sim}{W}$$

From the multiplication law above, we can see that

$$\frac{\vec{\alpha}\cdot\vec{\sigma}}{2} W - W \frac{\vec{\alpha}\cdot\vec{\sigma}}{2} = i\epsilon^{ijk}\alpha^i W^j \sigma^k$$

Then the infinitesimal motion

$$W \longrightarrow W - i \frac{\vec{\alpha}\cdot\vec{\sigma}}{2} W + i W \frac{\vec{\alpha}\cdot\vec{\sigma}}{2}$$

induces

$$W^i \rightarrow W^i + \epsilon^{ijk}\alpha^j W^k$$

which we recognize is just an infinitesimal rotation about the axis $\hat{\alpha}$. The infinitesimal motion above integrates to

$$W \rightarrow U W U^\dagger$$

This indicates that the formula

$$U = \exp\left[-i\,\alpha\,\hat{n}\cdot\frac{\vec{\sigma}}{2}\right]$$

does indeed give a representation of rotations. In this expression, vectors are represented by 2×2 Hermitian matrices, and rotations by 2×2 unitary matrices acting on the Hermitian matrices by

$$w \to w' = U w U^{-1}$$

Notice that w' is also, consistently, a traceless Hermitian matrix.

We can use the Cayley-Klein parametrization to get a clearer idea of the shape of the group of 3-dimensional rotations. Since

$$(\sigma^i)^2 = 1 \qquad (\hat{n}\vec{\sigma})^2 = 1$$

we can expand

$$\begin{aligned} e^{-i\vec{\alpha}\cdot\vec{\sigma}/2} &= 1 - i\,\frac{\vec{\alpha}\cdot\vec{\sigma}}{2} + \frac{1}{2}\frac{(-i\vec{\alpha}\cdot\vec{\sigma})^2}{2} + \cdots \\ &= 1 - i\,\vec{\alpha}\cdot\frac{\vec{\sigma}}{2} - \frac{1}{2}\left(\frac{\alpha}{2}\right)^2 + \cdots \\ &= \cos\left|\frac{\vec{\alpha}}{2}\right| - i\,\hat{\alpha}\cdot\vec{\sigma}\,\sin\left|\frac{\vec{\alpha}}{2}\right| \end{aligned}$$

so the matrix U can be written

$$U = \cos\frac{\alpha}{2} - i\sin\frac{\alpha}{2}\;\hat{\alpha}\cdot\vec{\sigma}$$

Since U is unitary, its inverse is

$$U^{\dagger}(\vec{\alpha}) = \cos\frac{\alpha}{2} + i\,\hat{\alpha}\cdot\vec{\sigma}\,\sin\frac{\alpha}{2} = U(-\vec{\alpha})$$

which is consistent. The eigenvalues of the matrix $\hat{\alpha}\cdot\vec{\sigma}$ are ±1; thus, U has eigenvalues

$$e^{\pm i\frac{\alpha}{2}}$$

This implies that U is a matrix of $SU(2)$. In fact, the elements of this form are the most general matrices of $SU(2)$. The group $SU(2)$ has $2^2 - 1 = 3$ generators, exactly like the group $SO(3)$. Since $SU(2)$ and $SO(3)$ have the same dimension near the identity and the same transformation law, these two groups are isomorphic in the neighborhood of the identity. However, the groups are not identical globally. The matrices U and $(-U)$ give the same tranformation

$$W \to U W U^{\dagger}$$

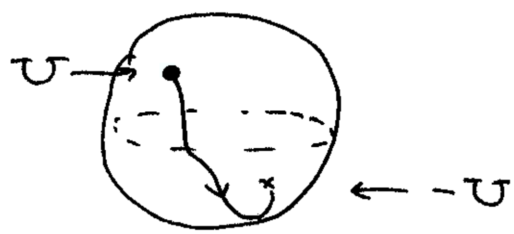
on 3-vectors. So, $SU(2)$ is a *double cover* of $SO(3)$; each $SO(3)$ element corresponds to *two* $SU(2)$ elements.

The unitary matrix U can also be written as

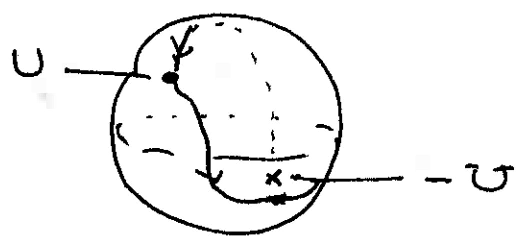
$$U = n^0 - i\,\vec{n}\cdot\vec{\sigma}$$

where $(n^0, \vec{n})$ is a unit vector in 4 dimensions. The matrices of $SU(2)$ are in 1-to-1 correspondence with points on the unit sphere in 4 dimensions (called S^3). The rotations in $SO(3)$ are in 1-to-2 correspondence with points on this sphere, with opposite points on the sphere giving the same rotation.

A consequence of this last statement is that $SO(3)$ contains closed paths that are topologically nontrivial. In particular, a 2π rotation corresponds to a path on the sphere of the form



A path through 4π can be contracted and is equivalent to a rotation through an angle of zero.



In classical mechanics, this topological aspect of the rotation group is just a curiosity. However, in quantum mechanics, it is of crucial importance. It implies that there is no difficulty in assigning a particle to pick up a phase factor (-1) under a 2π rotation. This feature allows the existence of particles in spinor (half-integer) representations of the rotation group. Eventually, it leads to the existence of particles with Fermi-Dirac statistics.