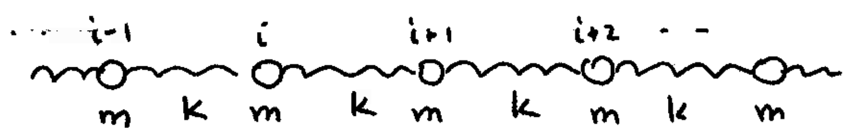
Nov. 30

Lagrangian Mechanics of Fields

Up to this point in the course, I have discussed only systems with a finite number of degrees of freedom. Many of the tools that we have developed are special to such systems. However, the general principles of mechanics apply to any mechanical system, including ones with continuum dynamical variables. In these last two lectures, I will discuss the generalization of Lagrangian mechanics to continuum systems.

To begin, I will derive a Lagrangian treatment of a simply continuum system by considering it as the limit of a system with discrete degrees of freedom. Earlier in the course, we considered a system of identical masses and springs laid out along the $\hat{x}$ axis



Let ξ_i be the displacement of the ith mass in the $\hat{x}$ direction. Then the Lagrangian of this system is

$$L = \sum_i \frac{1}{2} m \dot{\xi}_i^2 - \sum_i \frac{1}{2} k (\xi_i - \xi_{i+1})^2$$

Varying this Lagrangian with respect to ξ_j, we find the equation of motion

$$m \ddot{\xi}_i = -k(\xi_i - \xi_{i-1}) - k(\xi_i - \xi_{i+1})$$

We can imagine that this system is an approximation to a continuous distribution of masses along a string. To pass to the continuum limit, we decrease the separation of the point masses while keeping the total mass per unit length fixed. The mass density on the string is

$$\rho = m/\Delta$$

so we must take Δ and m to zero in fixed ratio. If we think if the displacement ξ_i as a continuous function of x, the finite difference $\xi_{i+1} - \xi_i$ is an approximation to the derivative of ξ with respect to x. More precisely,

$$\xi_{i+1} - \xi_i \approx \Delta \frac{d\xi}{dx}$$

$$2\xi_i - \xi_{i+1} - \xi_{i-1} \cong -\Delta^2 \frac{d^2\xi}{dx^2}$$

Then if we set

$$k = \frac{\kappa}{\Delta}$$

we obtain a set of equations that have a consistent form as $\Delta \to 0$.

$$\Delta \rho \cdot \ddot{\xi}_i = \frac{\kappa}{\Delta} \Delta^2 \frac{d^2\xi}{dx^2}$$

or, finally

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{\kappa}{\rho} \frac{\partial^2 \xi}{\partial x^2}$$

You recognize this as the wave equation in 1 dimension. Its solutions are

$$\xi(x,t) = f(x+ct) + g(x-ct)$$

where

$$c = \sqrt{\frac{K}{\rho}}$$

is the speed of the waves along the string. The form of c makes sense: If the masses are larger, the system will be more sluggish; if the springs are stiffer, the wave speed will be higher.

I will now rederive this equation directly from the Lagrangian. If we insert the formulae

$$m = \rho\Delta \qquad\qquad k = \frac{K}{\Delta}$$

into the Lagrangian above, we find

$$L = \sum_i \frac{1}{2}\Delta\rho\,\dot{\xi}^2 - \frac{1}{2}\frac{K}{\Delta}\left(\Delta\frac{\partial\xi}{\partial x}\right)^2$$

The difference Δ is the interval in x, so

$$\sum_i \Delta = \int dx$$

and we find

$$L = \int dx \left[\frac{1}{2}\rho\,\dot{\xi}^2 - \frac{1}{2}K\left(\frac{\partial\xi}{\partial x}\right)^2\right]$$

The action integral then takes the form

$$S = \int dt\,dx \left[\frac{1}{2}\rho \left(\frac{\partial \xi}{\partial t}\right)^2 - \frac{1}{2}\kappa \left(\frac{\partial \xi}{\partial x}\right)^2 \right]$$

Note that space and time come together neatly as an integral over spacetime.

If this is the action, we should be able to derive the equations of motion from the principle $\delta S = 0$. Indeed varying

$$\xi(x,t) \to \xi(x,t) + \delta\xi(x,t)$$

we find

$$\delta S = \int dt\,dx \left[\rho \frac{\partial \xi}{\partial t} \frac{\partial}{\partial t} \delta\xi - \kappa \frac{\partial \xi}{\partial x} \frac{\partial}{\partial x} \delta\xi \right]$$

Integral both terms by parts, and, just for a moment, ignore any contributions from the boundaries. Then we find

$$\delta S = \int dt\,dx \; \delta\xi(x,t) \left[\rho \frac{\partial^2}{\partial t} \xi - \kappa \frac{\partial^2}{\partial x^2} \xi \right]$$

The coefficient of the variation $\delta\xi(x)$ is indeed the equation of motion for disturbances on the string.

To complete this derivation, we need to discuss the boundary conditions. The integral over x is taken from $-\infty$ to ∞. In typical applications, nothing is happening at spatial infinity and we can place the boundary condition that fields go to zero there. In the example above, the spatial boundary term is

$$\int dt \left(-\kappa \frac{\partial \xi}{\partial x} \delta\xi \right) \Bigg|_{x=-\infty}^{x=\infty}$$

and so its vanishing requires only that

$$\frac{\partial \xi}{\partial x} \to 0 \quad \text{as} \quad |x| \to \infty$$

In the time direction, the story is different, but the correct statements are familiar from our discussion of particle mechanics. In systems with discrete degrees of freedom, we varied the coordinates $q_i(t)$ subject to fixed initial and final conditions

$$q_i(t_1) \ , \quad q_i(t_2) = \text{fixed}.$$

In this continuum situation, the coordinates are the values of the function $\xi(x,t)$, viewed as a function of x at fixed t. The initial and final conditions are then given by specifying $\xi(x,t)$ at an initial time t_1 and a final time t_2. The action integral is not taken over all of space-time but rather over a slice of spacetime between t_1 and t_2. In the derivation above, the boundary terms in the time direction are

$$\int dx \quad \rho \frac{\partial \xi}{\partial t} \ \delta \xi(x,t) \Big|_{t=t_1}^{t=t_2}$$

Since $\xi(x,t)$ is specified at $t=t_1$ and $t=t_2$,

$$\delta \xi(x,t_1) = \delta \xi(x,t_2) = 0$$

and these boundary terms vanish.

One further aspect of this variational principle deserves comment. In a system with discrete degrees of freedom, the action takes the form

$$S = \int dt\, L$$

In a continuum system, as illustrated by the example above, L is itself an integral over x, so that

$$S = \int dt\, dx\, \mathcal{L}$$

where $\mathcal{L}$ is a function of the fields and their derivatives at the spacetime point $(\vec{x}, t)$. This function is called the *Lagrange density.* The action is then of the form

$$S = \int dt\, dx \quad \mathcal{L}\left(\xi(\vec{x},t),\ \frac{\partial}{\partial t}\xi(\vec{x},t),\ \vec{\nabla}\xi(\vec{x},t) \right)$$

For a field $\xi(\vec{x}, t)$, the variation of this action, after integrations by parts, will fall naturally into the form

$$\delta S = \int dt\, dx \quad \delta\xi(\vec{x},t) \cdot \left[\text{function of } \xi + \text{derivs. at } (\vec{x},t) \right]$$

The quantity in braces, which is set to zero in the equation of motion, is then also a function of $\xi(\vec{x}, t)$ and its derivatives. By writing the action as the spacetime integral of a Lagrange density, we automatically obtain field equations that are *local* in space and time.

We have now developed the formalism that we need to make a complete generalization from Lagrangian particle mechanics to Lagrangian continuum mechanics.

In the particle case, the coordinates are $q_i(t)$, $i = 1, \ldots, n$, with

$$\delta q_i(t_1) = \delta q_i(t_2) = 0$$

at the initial and final times t_1 and t_2.

In the continuum case, the coordinates are $\phi_i(\vec{x}, t)$, $i = 1, \ldots, n$, with

$$\delta\phi_i(\vec{x}, t_1) = \delta\phi_i(\vec{x}, t_2) = 0$$

at the initial and final times t_1 and t_2.

In the particle case, the action integral is

$$S = \int dt \; L(q_i, \dot{q}_i)$$

computed over the interval $t = t_1$ and $t = t_2$.

In the continuum case, the action integral is

$$S = \int dt\, dx \; \mathcal{L}\left[\phi_i(\vec{x}, t), \dot{\phi}_i(\vec{x}, t), \frac{\partial}{\partial x_j}\phi_i(\vec{x}, t)\right]$$

computed over the interval $t = t_1$ and $t = t_2$ and over all $\vec{x}$, or over a definite region of $\vec{x}$ with proper boundary conditions at the spatial boundaries.

In the particle case, dynamical principle is

$$\delta S = 0$$

In the continuum case, dynamical principle is

$$\delta S = 0$$

In the particle case, it is convenient to define conjugate momenta

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

Then the equations of motion are the Euler-Lagrange equations

$$\frac{d}{dt} p_i = \frac{\partial L}{\partial q_i}$$

In the continuum case, it is convenient to define conjugate momentum densities

$$\pi_i = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i}$$

and also their counterparts with respect to spatial derivatives

$$\pi_i^{\,j} = \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial}{\partial x_j} \phi\right)}$$

Then the equations of motion generalize the Euler-Lagrange equations in a way that we will see in a moment.

Beginning with the action

$$S = \int dt\, dx\; \mathcal{L}\left[\phi_i, \frac{\partial}{\partial t}\phi_i, \frac{\partial}{\partial x_j}\phi_i\right]$$

apply the variational principle $\delta S = 0$.

$$\delta S = \int dt\,dx \left[\delta\phi_i \frac{\partial \mathcal{L}}{\partial \phi_i} + \frac{\partial}{\partial t}\delta\phi_i \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi_i}{\partial t}\right)} + \frac{\partial}{\partial x^j}\delta\phi_i \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi_i}{\partial x^j}\right)} \right]$$

Integrate by parts in the last two terms. The boundary conditions should be such that we can ignore the surface terms.

$$\delta S = \int dt\,dx \;\; \delta\phi_i(\vec{x},t) \left[\frac{\partial \mathcal{L}}{\partial \phi_i} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi_i}{\partial t}\right)} - \frac{\partial}{\partial x^j} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi_i}{\partial x^j}\right)} \right]$$

Then the equations of motion are a continuum generalization of the Euler-Lagrange equations

$$\frac{\partial}{\partial t}\pi_i + \frac{\partial}{\partial x^j}\pi_i^j = \frac{\partial \mathcal{L}}{\partial \phi_i}$$

I will now give some examples of the use of this formalism. First, consider the action for a complex-valued field $\psi(\vec{x},t)$

$$S = \int dt\,d^3x \left[\psi^* i \frac{\partial}{\partial t}\psi - \frac{1}{2m}(\vec{\nabla}\psi)^* \vec{\nabla}\psi - \psi^* V(x)\psi \right]$$

The action must be real-valued, and that is the case here, as long as we can ignore boundary terms in time when integrating by parts

$$\left(\int dt\,d^3x \;\; \psi^* i \frac{\partial}{\partial t}\psi \right)^* = \int dt\,d^3x \;\; \psi (-i) \frac{\partial}{\partial t}\psi^* = \int dt\,d^3x \frac{\partial \psi}{\partial t} \cdot i \cdot \psi^*$$

The variation of the action is

$$\delta S = \int dt\, d^3x \Big[\delta\psi^* \, i\frac{\partial}{\partial t}\psi + \psi^* i \frac{\partial}{\partial t}\delta\psi$$
$$- \frac{1}{2m}(\vec{\nabla}\delta\psi^* \vec{\nabla}\psi + \vec{\nabla}\psi^* \vec{\nabla}\delta\psi) - \delta\psi^* V\psi - \psi^* V\delta\psi \Big]$$

After integrating by parts,

$$\delta S = \int dt\, d^3x \Big\{ \delta\psi^* \Big[i\frac{\partial}{\partial t}\psi + \frac{1}{2m}\nabla^2\psi - V\psi \Big]$$
$$+ \Big[-i\frac{\partial}{\partial t}\psi^* + \frac{1}{2m}\nabla^2\psi^* - \psi^* V \Big] \delta\psi \Big\}$$

The equation of motion is then

$$i\frac{\partial}{\partial t}\psi = -\frac{1}{2m}\nabla^2\psi + V(x)\psi$$

and its complex conjugate. This is the Schrödinger equation with $\hbar = 1$, viewed as a classical partial differential equation.

In the wave equation and in the general Euler-Lagrange equation for fields, time and space derivatives appear in a symmetrical way. This means that it should be straightforward to set up a Lagrangian formalism for fields that is manifestly invariant under the transformations of special relativity. To do this most easily, I will now set up a relativistic notation that is uniform between space and time. Write the spacetime coordinates as

$$x^\mu = (ct, \vec{x})$$

where Roman indices $i, j, \ldots$ run over the values $1, 2, 3$ and Greek indices $\mu, \nu, \ldots$ run over four values $0, 1, 2, 3$. Measure time in cm by writing

$$x^0 = ct$$

The derivatives with respect to spacetime coordinates are written

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x^i}\right)$$

Introduce a metric tensor

$$g^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \qquad g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

and use this tensor to raise and lower indices,

$$x_\mu = g_{\mu\nu} x^\nu \qquad \partial^\mu = g^{\mu\nu} \partial_\nu$$

$$= (ct, -\vec{x}) \qquad = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\frac{\partial}{\partial x^i}\right)$$

The combination

$$x^\mu g_{\mu\nu} y^\nu = x^\mu y_\mu$$

is the invariant product of two 4-vectors

$$x^0 y^0 - \vec{x} \cdot \vec{y}$$

A Lorentz transformation $\Lambda^{\mu}{}_{\nu}$ is a linear transformation that leaves this expression invariant

$$x' = \Lambda x \qquad y' = \Lambda y \qquad \rightarrow \qquad x'^{\mu} y'_{\mu} = x^{\mu} y_{\mu}$$

This implies

$$g_{\mu\sigma} \, \Lambda^{\mu}{}_{\nu} \, \Lambda^{\sigma}{}_{\rho} = g_{\nu\rho}$$

or

$$\Lambda^T g \Lambda = g \qquad \Rightarrow \qquad \Lambda^{-1} = g \Lambda^T g$$

The derivative transforms as

$$\frac{\partial}{\partial x'^{\mu}} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \frac{\partial}{\partial x^{\nu}} \quad \text{or} \quad \frac{\partial}{\partial x'} = \Lambda^{-1} \frac{\partial}{\partial x}$$

Then, the following combinations are also invariant under Lorentz transformations

$$x^{\mu} \partial_{\mu} \qquad\qquad \partial^{\mu} \partial_{\mu} = \partial_{\mu} g^{\mu\nu} \partial_{\nu}$$

The first of these justifies the convention I made above in defining the derivative with a lowered index. The second is the familiar d'Alembertian or relatistic wave equation operator

$$\partial^\mu \partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

One often denotes

$$\Box = \partial^\mu \partial_\mu$$

Using the same method as in the lecture on Lie groups, we can count the number of independent Lorentz transformations. Consider an infinitesimal Lorentz transformation

$$\Lambda = 1 + K$$

The relation

$$\Lambda^{-1} = g\,\Lambda^T g$$

gives

$$1 - K = g(1 + K^T)g \Rightarrow -gK = K^T g$$
$$\Rightarrow (gK) = -(gK)^T$$

Then gK must be a 4×4 antisymmetric matrix. Such a matrix has 6 components. Rotations in 3 dimensions preserve the Lorentz inner product, and there are 3 independent 3-dimensional rotations. The other three infinitesimal elements are infinitesimal boosts in the $\hat{x}$, $\hat{y}$ and $\hat{z}$ directions.

It is not difficult to find an action $S[\phi]$ which leads to the relativistic wave equation $\partial^\mu \partial_\mu \phi = 0$ as its variational equation. Try

$$S = \int dt\, d^3x \; \tfrac{1}{2}\, \partial^\mu \varphi\, \partial_\mu \varphi$$

Then

$$\delta S = \int dt\, d^3x \; \partial^\mu \delta\varphi\, \partial_\mu \varphi$$

$$= \int dt\, d^3x \; \delta\varphi(\vec{x},t)\,[-\partial^\mu \partial_\mu \varphi\,]$$

Then

$$\delta S = 0 \qquad \Rightarrow \qquad \partial^\mu \partial_\mu \varphi = 0$$

Notice that the action that we have just written is invariant to Lorentz transformations. The Lagrange density is manifestly invariant. This is automatically guaranteed by the matching of raised and lowered Lorentz indices. The integration measure transforms as

$$dt\, d^3x \to dt'\, d^3x' = \det \Lambda \cdot dt\, d^3x$$

For an infinitesimal Lorentz transformation, the diagonal elements of K are zero. Thus

$$\det(1+K) = 1 + \mathrm{tr}\, K = 1 \quad \text{up to} \quad \mathcal{O}(K^2)$$

Integrating this up to a finite transformation

$$\det \Lambda = 0$$

So, the integration measure also satisfies

$$dt\, d^3x = dt'\, d^3x'$$

and the complete action integral is invariant.

In a relativistic theory, the conjugate momentum densities are unified into a single object. Let

$$\Pi_i^\mu = \frac{\partial \mathcal{L}}{\partial\, \partial_\mu \varphi_i}$$

where

$$\Pi_i^0 = c\, \Pi_i \qquad \Pi_i^j = \Pi_i^j$$

Then the Euler-Lagrange equation takes the form

$$\partial_\mu \Pi_i^\mu = \frac{\partial \mathcal{L}}{\partial \phi_i}$$

For the system just described

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = \partial^\mu \phi$$

so we find in another way that the variational equation is

$$\partial_\mu \partial^\mu \phi = 0$$

The slightly more complicated Lagrange density

$$\mathcal{L} = \frac{1}{2} (\partial^\mu \phi)^2 - \frac{1}{2} m^2 \phi^2$$

gives

$$S = \int dt\, d^3x \left[\frac{1}{2} \partial^\mu \phi\, \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 \right]$$

The Euler-Lagrange equation is the *Klein-Gordon equation*

$$\partial^\mu \partial_\mu \phi + m^2 \phi = 0$$

To study interacting fields, one can add a nonlinear term to the Lagrange density

$$\mathcal{L} = \frac{1}{2} \partial^\mu \phi\, \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4} \phi^4$$

The resulting theory is called ϕ^4 *theory.* Its equation of motion is

$$\partial^\mu \partial_\mu \phi + m^2 \phi + \lambda \phi^3 = 0$$

This theory has applications in particle physics and in the theory of second-order phase transitions.

A very important case of a relativistic field theory is that of the electromagnetic field. Here is its relativistic description: The basic coordinates are a set of 4 fields, the 4-component electromagnetic vector potential

$$A^\mu = (\phi, A^i)$$

The 0 component $\phi(x)$ is the familiar scalar potential. The electromagnetic field strength is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

This tensor is antisymmetric under $\mu \leftrightarrow \nu$. Its two cases are

$$F_{0i} = \partial_0 A_i - \partial_i A_0$$

$$F_{ij} = \partial_i A_j - \partial_j A_i$$

Remember that

$$A_i = -A^i$$

then

$$F_{0i} = -\frac{1}{c}\frac{\partial}{\partial t}A^i - \frac{\partial}{\partial x^i}\phi = E^i$$

$$F_{ij} = -\partial_i A^j - \partial_j A^i = -\epsilon^{ijk} B^k$$

I claim that the appropriate action for this system is

$$S = \int d^4x \left\{ -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - A_\mu j^\mu \right\}$$

where

$$j^\mu = \left(\rho(x), \frac{1}{c} j^i(x) \right)$$

is the 4-vector of electric charge density and electric current To find the field equations that result from this action, carry out the variation

$$\delta S = \int d^4x \left\{ -\frac{1}{8\pi}(\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu) F^{\mu\nu} - \delta A_\mu j^\mu \right\}$$

Since $F_{\mu\nu}$ is antisymmetric in its indices, the two terms in parentheses are equal. So this becomes, after integration by parts,

$$\delta S = \int d^4x \;\; \delta A_\mu(x) \left\{ -\frac{1}{4\pi}\partial_\nu F^{\mu\nu} - j^\mu \right\}$$

The variational equation of motion is

$$\partial_\mu F^{\mu\nu} = 4\pi j^\nu$$

To understand this equation, write the two cases $\nu = 0$ and $\nu = i$ more explicitly. For $\nu = 0$,

$$\partial_0 F^{00} + \partial_i F^{i0} = 4\pi j^0$$

$$0 + \partial_i E^i = 4\pi\rho$$

so this equation is just

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$$

For $\nu = i$, the equation is

$$\partial_0 F^{0i} + \partial_j F^{ji} = 4\pi j^i/c$$

$$-\frac{1}{c}\frac{\partial}{\partial t}E^i + \frac{\partial}{\partial x^j}\epsilon^{ijk}B^k = 4\pi j^i/c$$

so this equation is

$$\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial}{\partial t}\vec{E} = \frac{4\pi}{c}\vec{j}$$

Thus, the equation

$$\partial_\mu F^{\mu\nu} = 4\pi j^\nu$$

is precisely a unified form of the two inhomogeneous Maxwell equations. The homogeneous Maxwell equations follow from the definition of $F_{\mu\nu}$ in terms of A_μ above, which implies the identity

$$\epsilon^{\nu\alpha\beta\gamma} \partial_\alpha F_{\beta\gamma} = 0$$

The case $\nu = 0$ is

$$\epsilon^{ijk} \partial_i F_{jk} = 0 \quad \Rightarrow \quad \vec{\nabla} \cdot \vec{B} = 0$$

The case $\nu = i$ is

$$-\epsilon^{ijk} \frac{1}{c}\frac{\partial}{\partial t} F_{jk} + \epsilon^{ijk} \frac{\partial}{\partial x^j} F_{0k} - \epsilon^{ijk} \frac{\partial}{\partial x^j} F_{k0} = 0$$

or

$$2 \frac{1}{c}\frac{\partial}{\partial t} B^i + 2\epsilon^{ijk} \nabla^j E^k = 0$$

$$\frac{1}{c}\frac{\partial}{\partial t} \vec{B} + \vec{\nabla} \times \vec{E} = 0$$

This formulation of the Maxwell equations has one more pretty feature. Since $F_{\mu\nu}$ is antisymmetric, it is obvious that

$$\partial_\nu (\partial_\mu F^{\mu\nu}) = 0$$

This implies

$$\partial_\nu j^\nu = 0$$

that is

$$\frac{\partial}{\partial t}\rho + \vec{\nabla}\cdot\vec{j} = 0$$

the conservation of the electric current. Thus, any source of the electromagnetic field must be a conserved current. Maxwell used current conservation to find the correct form of Ampere's law to complete the original derivation of his equations. It is pleasing to see that, in this highly symmetry form of the Maxwell theory, current conservation is incorporated very elegantly.