Dec. 2

Symmetries and Conservation Laws in Field Theory

In the previous lecture, I explained how Lagrangian formalism extends to continuum systems. We saw how to use variational principles to derive equations of motion for fields. In particular, we studied the example of the electromagnetic field. We saw that the Lagrangian approach incorporates the idea that field equations are local in spacetime. We also saw that it is easy to write field equations invariant under Lorentz transformations by starting from a Lorentz-invariant action principle.

The Lagrangian formalism for continuum systems also offers a generalization of Noether's theorem that each continuuous symmetry of the Lagrangian gives a corresponding conservation law.

To discuss this point, we should recall how Noether's theorem is proved for systems with a finite number of degrees of freedom. We consider infinitesimal transformations of the coordinates

$$q_i \longrightarrow q_i + \alpha \, \Delta q_i(q, \dot{q})$$

where α is an infinitesimal parameter. This transformation is a symmetry if it leaves the action unchanged, up to boundary terms that do not affect the location of the extremal paths. This requires

$$L \rightarrow L + \alpha \frac{d}{dt} F(q, \dot{q})$$

To prove Noether's theorem, we generalized α to a parameter $\delta\alpha(t)$ with arbitrary time-dependence. We computed the variation of the action under this more general variation of the coordinates, and we obtained

$$\delta S = \int dt \;\, \delta\alpha(t) \left[-\frac{d}{dt}(p_i \Delta q_i) + \frac{d}{dt} F \right]$$

The Lagrangian principle $\delta S = 0$ then implies the conservation law

$$\frac{d}{dt}\left[p_i \Delta q_i - F \right] = 0$$

Here is the analogous argument for a continuum Lagrangian system. For such a system, the Lagrangian is written in terms of a local Lagrange density as

$$S = \int d^4x \; \mathcal{L}[\phi_i, \dot{\phi}_i, \frac{\partial}{\partial x^j}\phi_i]$$

An infinitesimal transformation of the fields can be written

$$\phi_i(x) \rightarrow \phi_i(x) + \alpha \, \Delta\phi_i$$

If this is a symmetry, it must leave the action invariant up to a surface term, which might be at the temporal or at the spatial boundary. Then the Lagrange density must at worst transform as

$$\mathcal{L} \rightarrow \mathcal{L} + \alpha \, \partial_\mu \mathcal{J}^\mu(\phi)$$

Now generalize the constant parameter α to a parameter $\delta\alpha(x)$ with arbitrary dependence on the spacetime point x. Varying the fields by

$$\delta\phi_i = \delta\alpha(x) \, \Delta\phi_i$$

changes the Lagrange density by the term found previously, plus new terms that involve the derivatives of the field

$$\int d^4x \quad \partial_\mu \delta\alpha \quad \Delta\phi_i \cdot \frac{\partial \mathcal{L}}{\partial\, \partial_\mu \phi_i}$$

The variation of the action is then

$$\delta S = \int d^4x \quad \delta\alpha(x) \left[- \partial_\mu \Pi^\mu_i \Delta\phi_i + \partial_\mu \mathcal{J}^\mu \right]$$

The principle $\delta S = 0$ then gives the equation

$$\partial_\mu J^\mu = 0$$

where

$$J^\mu = \Pi_i^\mu \Delta\phi_i - \mathcal{J}^\mu$$

This equation is a conservation law, and more. First of all, it implies the existence of a global conservation law. Integrating this equation over space,

$$0 = \int d^3x \;\; \partial_\mu J^\mu = \int d^3x \left[\frac{1}{c}\frac{d}{dt} J^0 + \vec{\nabla} \cdot \vec{J} \right]$$

The second term under the integral is a surface term at the spatial boundary. If $\vec{J}$ goes to zero at infinity,

$$0 = \int d^3x \;\; \frac{d}{dt} J^0$$

Then

$$\frac{d}{dt} Q = 0 \qquad \text{where} \qquad Q = \int d^3x \, J^0$$

However, the equation we have derived is much stronger than that. The object J^μ is a 4-vector current. Writing it in components as we did for the electromagnetic current in the previous lecture,

$$J^\mu = \left(\rho, \frac{1}{c}\vec{j}\right)$$

the equation for J^μ becomes

$$\frac{\partial}{\partial t}\rho + \vec{\nabla}\cdot\vec{j} = 0$$

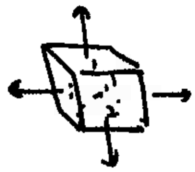
This is a *local conservation law*, expressing the conservation of the density ρ in each small region of space. Integrating the equation over a small volume V, we find

$$\frac{\partial}{\partial t}\int_V d^3x \, \rho = -\int_V d^3x \, \vec{\nabla}\cdot\vec{j}$$

that is

$$\frac{d}{dt}\int_V d^3x \, \rho = -\int_{\partial V} d^2S \, \hat{n}\cdot\vec{j}$$

The change in the charge Q over the small region is compensated by the flow of current through the walls that surround the region.



In the same way that Lagrangian field theory naturally gives us local equations of motion, it gives us local conservation laws.

As a first example, consider the Lagrangian for the Schrödinger equation discussed in the previous lecture,

$$\mathcal{L} = \psi^* i \frac{\partial}{\partial t} \psi - \frac{1}{2m} \vec{\nabla}\psi^* \vec{\nabla}\psi - \psi^* V(x) \psi$$

This Lagrangian has the symmetry

$$\psi \to e^{i\alpha} \psi \qquad\qquad \psi^* \to \psi^* e^{-i\alpha}$$

or, in infinitesimal form

$$\Delta\psi = i\alpha\psi \qquad\qquad \Delta\psi^* = -i\alpha\psi^*$$

Under this transformation, the Lagrange density is invariant, so that

$$\mathcal{J}^\mu = 0$$

Now vary the action with respect to the generalization of this transformation with a spacetime-dependent parameter $\delta\alpha(\vec{x}, t)$

$$\delta\psi = i\,\delta\alpha(x)\,\psi \qquad \delta\psi^* = -i\,\delta\alpha(x)\,\psi^*$$

Under this variation

$$\delta S = \int d^4x \Big\{ (-i\,\delta\alpha\,\psi^*)\, i\frac{\partial}{\partial t}\psi + \psi^*\, i\frac{\partial}{\partial t}(i\,\delta\alpha\,\psi) - \frac{1}{2m}\left[\vec\nabla(-i\,\delta\alpha\,\psi^*)\vec\nabla\psi + \vec\nabla\psi^*\,\vec\nabla(i\,\delta\alpha\,\psi)\right] - (-i\,\delta\alpha\,\psi^* V\psi + \psi^* V\, i\,\delta\alpha\,\psi) \Big\}$$

All terms cancel except for those in which a derivative acts on $\delta\alpha$,

$$\delta S = \int d^4x \Big\{ -\delta\dot\alpha\,(\psi^*\psi) + \frac{i}{2m}(\vec\nabla\,\delta\alpha)\left[\psi^*\vec\nabla\psi - (\vec\nabla\psi^*)\psi\right] \Big\}$$

Finally, after integrating by parts,

$$\delta S = \int d^4x\ \delta\alpha(x) \Big\{ \frac{\partial}{\partial t}\psi^*\psi - \vec\nabla\left[\frac{i}{2m}\left(\psi^*\vec\nabla\psi - \vec\nabla\psi^*\psi\right)\right] \Big\}$$

Thus, we find the conservation law

$$\frac{\partial}{\partial t}\rho + \vec\nabla\vec{j} = 0$$

where the charge density $\rho(\vec{x},t)$ is

$$\rho(\vec{x},t) = \psi^*\psi$$

and the current $\vec{j}(\vec{x},t)$ is

$$\vec{j}(\vec{x},t) = \frac{-i}{2m}\left[\psi^* \vec{\nabla}\psi - \vec{\nabla}\psi^* \psi\right]$$

You will recognize the charge density as the probability density for the Schrödinger wavefunction. The current takes the form that is standard for the probability current. If the Schrödinger equation is used to describe an electron, the electric charge density and current of the electron are proportional to ρ and $\vec{j}$.

In a relativistic field theory, the analogous argument will produce a current that is manifestly a Lorentz 4-vector. Consider, for example, the theory of a complex-valued Klein-Gordon field,

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi$$

This theory has the global symmetry

$$\delta\phi = i\alpha\phi \qquad \delta\phi^* = -i\alpha\phi^*$$

with, again, $\delta\mathcal{L} = 0$. The variation of the action under the local version of this symmetry gives

$$\delta S = \int d^4x \left\{ \partial_\mu(-i\delta\alpha\phi^*)\partial^\mu\phi + \partial_\mu\phi^*\partial^\mu(i\delta\alpha\phi) - m^2 (-i\delta\alpha\phi^*)\phi + \phi^*(i\delta\alpha\phi) \right\}$$

or

$$\delta S = \int d^4x \, (\partial_\mu \delta\alpha) \left[-i\left(\phi^*\partial^\mu\phi - \partial^\mu\phi^*\phi\right)\right]$$

We then identify the charge-current 4-vector as

$$J^\mu = i\left[\phi^* \partial^\mu \phi - \partial^\mu \phi^* \phi\right]$$

and, indeed, this quantity is manifestly a 4-vector field.

In particle mechanics, the invariance of the action with respect to translations in space, translations in time, and rotations implies the basic spacetime conservation laws of momentum, energy, and angular momentum. In continuum mechanics, this statement is also correct, with the additional feature that these conservation laws are realized *locally*, with densities and currents of the conserved quantities.

I will first give a general treatment for the symmetries of space and time translation. In field theory, we can address these symmetries in a unified way. A translation sends a field configuration to a shifted one

$$\phi_i(x) \to \phi_i(x+a)$$

The infinitesimal form of this transformation is

$$\phi_i(x) \to \phi_i(x) + a^\nu \partial_\nu \phi_i(x)$$

If the Lagrange density $\mathcal{L}$ has no explicit dependence on $\vec{x}$ and t, it transforms similarly, as

$$\mathcal{L} \to \mathcal{L} + a^\nu \partial_\nu \mathcal{L}$$

Then, for a spacetime translation by a^ν

$$\Delta\phi_i = a^\nu \partial_\nu \phi_i \qquad \mathcal{J}^\nu = a^\nu \mathcal{L}$$

Generalizing this to a spacetime-dependent variation of the form

$$\phi_i \to \phi_i + n^\nu \, \delta a(x) \, \partial_\nu \phi_i$$

and using the formula for J^μ above, we find

$$J^\mu = \Pi_i^\mu \, n^\nu \partial_\nu \phi_i - n^\mu \mathcal{L}$$

Extracting the fixed vector n^ν that gives the direction of the translation, we find

$$J^\mu = \Theta^{\mu\nu} n_\nu$$

where

$$\Theta^{\mu\nu} = \Pi_i^\mu \partial^\nu \phi_i - g^{\mu\nu} \mathcal{L}$$

This is a 4-vector of conserved currents, more properly, a *tensor*. It is called the *canonical energy-momentum tensor*.

The conservation of $\Theta^{\mu\nu}$ leads to a set of 4 global conservation laws

$$\partial_\mu \Theta^{\mu\nu} = 0 \Rightarrow \frac{d}{dt}\int d^3x \, \Theta^{00} = 0 \, , \quad \frac{d}{dt}\int d^3x \, \Theta^{0i} = 0$$

The first of these, the conservation of a scalar quantity, can be associated with energy conservation. The second, the conservation of a 3-vector quantity, is the conservation of momentum. Then

Θ^{00} is the density of energy

$\frac{1}{c}\Theta^{0i}$ is the density of momentum

The local equation for $\nu = 0$

$$\frac{\partial}{\partial t}\Theta^{00} + \nabla^{i}(c\Theta^{i0}) = 0$$

then relates an energy density Θ^{00} to an energy current or flux

$$j^{i}_{\varepsilon} = c\,\Theta^{i0}$$

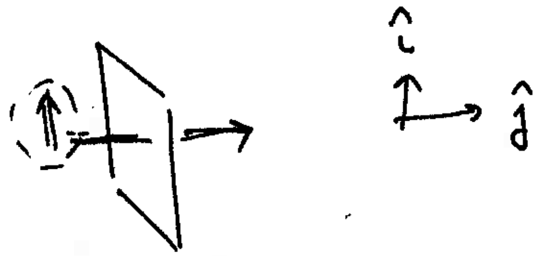
Similarly, the equation

$$\frac{\partial}{\partial t}\left(\frac{1}{c}\Theta^{0i}\right) + \nabla^{j}(\Theta^{ji}) = 0$$

relates the density of momentum in the $\hat{\imath}$ direction to the current of momentum in the $\hat{\imath}$ direction. The quantity

$$\Theta^{ji}$$

gives the flux of $\hat{\imath}$ momentum across a wall perpendicular to the $\hat{\jmath}$ direction



This quantity Θ^{ji} is called the *stress tensor.*

As a first example, I will work out the canonical energy-momentum tensor for the Klein-Gordon field. The Lagrange density is

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$$

For this system

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = \partial^\mu \phi$$

Then

$$\Theta^{\mu\nu} = \partial^\mu \phi \, \partial^\nu \phi - \frac{g^{\mu\nu}}{2} \left(\partial_\lambda \phi \, \partial^\lambda \phi - m^2 \phi^2 \right)$$

The energy density is

$$\Theta^{00} = \partial^0 \phi \, \partial^0 \phi - \frac{1}{2} \left(\partial^0 \phi \, \partial^0 \phi - \vec{\nabla} \phi \cdot \vec{\nabla} \phi - m^2 \phi^2 \right)$$

which conveniently reduces to

$$\Theta^{00} = \frac{1}{2} (\partial^0 \phi)^2 + \frac{1}{2} (\vec{\nabla} \phi)^2 + \frac{1}{2} m^2 \phi^2$$

this is quite a reasonable expression for the energy density of a field. It acquires contributions from the motion and from the spatial gradients of the field, and from the field potential energy

$$V = \int d^3x \ \frac{1}{2} m \phi^2$$

The momentum density of the field is given simply (since $g^{0i} = 0$) by

$$\Theta^{0i} = \partial^0 \phi \, \partial^i \phi$$

Notice that, up to factors of c, the energy-momentum tensor is *symmetric*. In particular, the momentum *density* equals the energy *current*,

$$\Theta^{0i} = \Theta^{i0}$$

Next, we can work out the energy-momentum tensor of the electromagnetic field. I will carry out the analysis with $j^\mu(x)0$. Then we will have the energy and momentum carried by the electromagnetic field itself. With a little more effort, we can compute the energy-momentum tensor for the electromagnetic field interacting with particles or fields. As long as the Lagrange density is free of explicit dependence on the spacetime point x, the Lagrangian dynamics will have a conserved energy and momentum. Energy and momentum can be exchanged between the field and its sources, but the total quantities will be unchanged and will be regulated by local conservation laws.

The Lagrangian of the pure electromagnetic field is

$$\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}$$

The momentum density conjugate to A^λ is

$$\frac{\partial \mathcal{L}}{\partial \partial_\mu A_\lambda} = -\frac{1}{4\pi} F^{\mu\lambda}$$

Then, using the formulae above, we fine

$$\Theta^{\mu\nu} = -\frac{1}{4\pi} F^{\mu\lambda} \partial^\nu A_\lambda + \frac{g^{\mu\nu}}{16\pi} F_{\lambda\sigma} F^{\lambda\sigma}$$

You can check that this expression is conserved. However, it is quite awkward. First, it is not gauge-invariant; it involves the A^μ field explicitly while physical observables of the electromagnetic field involve only the $\vec{E}$ and $\vec{B}$ fields. Second, it is not symmetric in $\mu \leftrightarrow \nu$.

Both problems with $\Theta^{\mu\nu}$ are cured using a trick due to Belinfante. Notice that, if $\Xi^{\mu\lambda\nu}$ is antisymmetric under $\mu \leftrightarrow \lambda$, then the tensor

$$B^{\mu\nu} = \partial_\lambda \Xi^{\mu\lambda\nu}$$

automatically satisfies

$$\partial_\mu B^{\mu\nu} = 0$$

Then this *Belinfante tensor* can be added to $\Theta^{\mu\nu}$ to produce a new conserved energy-momentum tensor that might have better properties. In this example, it is convenient to choose

$$\Xi^{\mu\lambda\nu} = \frac{1}{4\pi} F^{\mu\lambda} A^\nu$$

Then

$$B^{\mu\nu} = \frac{1}{4\pi}\partial_\lambda (F^{\mu\lambda} A^\nu) = \frac{1}{4\pi}(\partial_\lambda F^{\mu\lambda}) A^\nu + \frac{1}{4\pi} F^{\mu\lambda}(\partial_\lambda A^\nu)$$

The source-free Maxwell equations imply $\partial_\lambda F^{\mu\lambda} = 0$, so this is equivalent to

$$B^{\mu\nu} = \frac{1}{4\pi} F^{\mu\lambda} \partial_\lambda A^\nu$$

Then the new conserved energy-momentum tensor is

$$T^{\mu\nu} = \frac{1}{4\pi} F^{\mu\lambda}(\partial_\lambda A^\nu - \partial^\nu A_\lambda) + \frac{g^{\lambda\nu}}{16\pi} F_{\lambda\sigma} F^{\lambda\sigma}$$

which simplifies to

$$T^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\lambda} F_\lambda{}^\nu + \frac{g^{\mu\nu}}{4} F_{\lambda\sigma} F^{\lambda\sigma}\right)$$

The components of this tensor take familiar forms. To compute them, we need

$$\begin{aligned} \frac{1}{4} F_{\lambda\sigma} F^{\lambda\sigma} &= \frac{1}{4}\left(F_{0i} F^{0i} + F_{i0} F^{i0} + F_{ij} F^{ij}\right) \\ &= -\frac{2}{4}(E^i)^2 + \frac{1}{4}\epsilon_{ijk} B^k \epsilon^{ijl} B^l \\ &= -\frac{1}{2} E^2 + \frac{1}{2} B^2 \end{aligned}$$

Then the energy density associated with the new tensor is

$$T^{00} = \frac{1}{4\pi}\left(F^{0i} F_i{}^0 - \frac{1}{2} E^2 + \frac{1}{2} B^2\right)$$

which simplifies to

$$T^{00} = \frac{1}{8\pi}\left(E^2 + B^2\right).$$

This is of course the familiar form of the energy of the electromagnetic field. The momentum density is

$$T^{0i} = \frac{1}{4\pi} F^{0j} F_j{}^i$$

$$= \frac{1}{4\pi}(-E^j)(+\epsilon^{jik} B^k)$$

or, finally

$$T^{0i} = \frac{1}{4\pi}(\vec{E} \times \vec{B})^i$$

Since the new tensor is symmetric, the momentum density equals the energy flux, and both are given by the Poynting vector.

For completeness, we can work out the electromagnetic stress tensor. This is

$$T^{ij} = \frac{1}{4\pi}\left\{F^{i0} F_0{}^j + F^{ik} F_k{}^j - \left(-\tfrac{1}{2}E^2 + \tfrac{1}{2}B^2\right)\delta^{ij}\right\}$$

$$= \frac{1}{4\pi}\left\{-E^i E^j - \epsilon^{ik\ell}\epsilon^{kjm} B^\ell B^m + \tfrac{1}{2}E^2\delta^{ij} - \tfrac{1}{2}B^2\delta^{ij}\right\}$$

or

$$T^{ij} = \frac{1}{4\pi}\left[\left(\tfrac{1}{2}E^2\delta^{ij} - E^i E^j\right) + \left(\tfrac{1}{2}B^2\delta^{ij} - B^i B^j\right)\right]$$

It is wonderful that there is a trick that can make the energy-momentum tensor of the electromagnetic field symmetry, but this trick looks very special to electromagnetism. Is it alway possible, for any system, to find a symmetric energy-momentum tensor? It turns out that this question can be answered by looking into the coupling of the mechanical system to gravity. In Einstein's theory of gravity, the dynamical variable is the metric $g_{\mu\nu}$ of spacetime. The source of the gravitational field is the $T^{\mu\nu}$ of matter. The expression of the coupling of matter is gravity is that the variation of the matter action with respect to the gravitational field has the form

$$\delta S = \int d^4x \sqrt{g} \; \tfrac{1}{2}\delta g_{\mu\nu} T^{\mu\nu}$$

Just as we saw for the electromagnetic field in the previous lecture, the field equations of gravity insure that $T^{\mu\nu}$ defined in this way is conserved. Here is an example. The generally covariant action for electromagnetism is

$$S = \int d^4x \sqrt{g} \left(-\frac{1}{16\pi} g^{\mu\lambda} g^{\nu\sigma} F_{\mu\nu} F_{\lambda\sigma} \right)$$

where

$$\sqrt{g} = (-\det g)^{\frac{1}{2}}$$

Varying with respect to $g^{\mu\nu}$, we find

$$\delta S = \int d^4x \sqrt{g} \; \tfrac{1}{2}\delta g^{\alpha\beta} \; \frac{1}{4\pi}\left(F_{\alpha\nu} F^{\nu}{}_{\beta} + \frac{g_{\alpha\beta}}{4} F_{\mu\nu} F^{\mu\nu} \right)$$

so that the symmetric form of the energy-momentum tensor appears automatically.

There is much more to say about the mechanics of fields, but that material is for other, more advanced, courses. I hope that this course has given you a firm foundation for your further study.