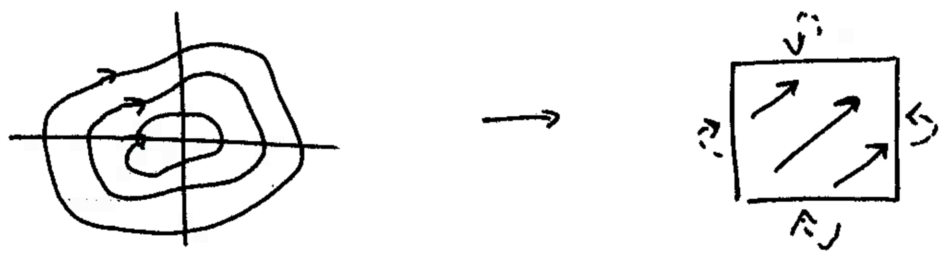
Nov. 11

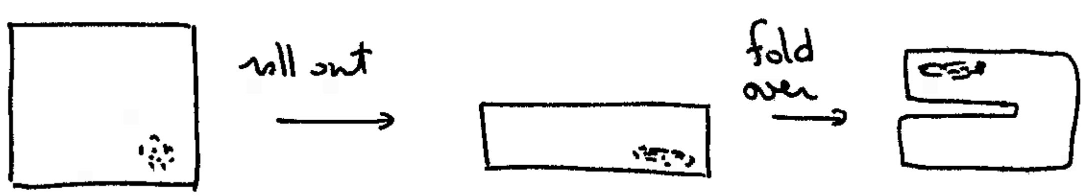
Regularity and Chaos in Mechanics

In the examples of Hamiltonian mechanics that we have discussed so far, the flow in phase space is intrinsically simple. In particular, for systems that can be converted to action and angle variables, the Hamiltonian flow is reduced to nothing more than translation on a torus,

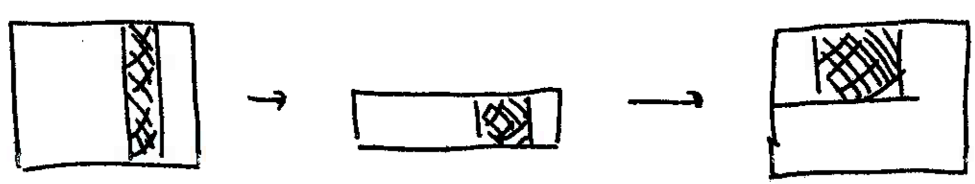


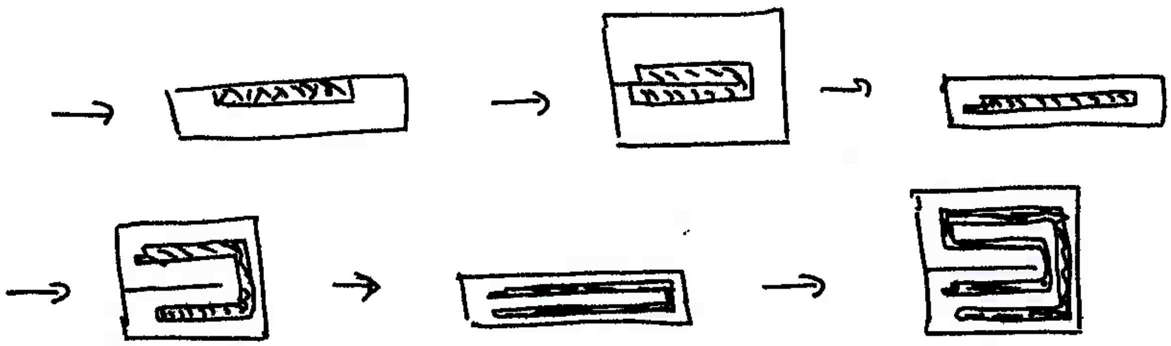
It is true that most of the Hamiltonian systems we have studied are energy-conserving system with one degree of freedom, that is, only to phase space coordinates. In such a system, with the restriction that phase space trajectories cannot cross, there is simply not room for interesting behavior.

Even in more general Hamiltonian systems, we have the constraint of conservation of areas in phase space. You might think that this constraint would severely constrain the possible motions. This is true, but nevertheless, the flows can be complex. A very simple example is given by the *Baker's transformation*:

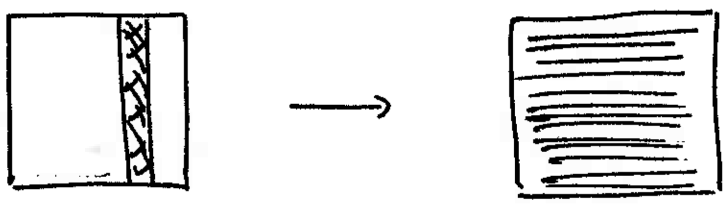


The basic operation is that used to make pastry dough, hence the name. The final shape is a square, and areas are conserved. Nevertheless, here is the result of carrying out a large number of Baker's transformations:





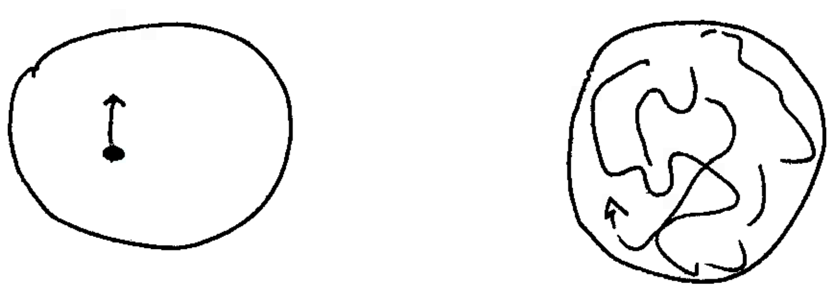
In the limit of an infinite number of transformations, the original marked region is dispersed throughout the square.



More rigorously, for any ϵ, there is an N such that, after N or more transformations, any point in the square is within a distance ϵ of some part of the shaded region. The image of the original shaded region then becomes *dense* in the full phase space.

Does this sort of extreme mixing actually happen in mechanical systems? Historically, there have been two different points of view. The first, very nineteenth century, point of view states that we could find conservation laws that make any Hamiltonian system integrable, if only we were clever enough. Then, for any system, there would exist a transformation that would straighten out and simplify all of the trajectories.

However, there is another nineteenth century idea that goes in exactly the opposite direction. Boltzmann and the other founders of statistical mechanics were impressed that systems of very many degrees of freedom rapidly evolve from simple initial conditions to complex and perhaps generic configurations. They introduced the idea that the time average of the motion of a complex system would look like an average over the surface of constant energy. This is formalized in the notion of *ergodicity*. An *ergodic* system is one in which a generic trajectory eventually comes close to every point on the surface of constant energy.



This idea justifies a statistical description of a system of many particles. We observe that a system with 10^{23} particles quickly attains a time-independent state that we identify with *thermal equilibrium.* In statistical mechanics, we represent thermal equilibrium by an average over points in phase space. If the system in question is ergodic, the only correct average would be one that includes all points in phase space. For such an average, the measure

$$\int dq\, dp \quad = \quad \int dq(t)\, dp(t)$$

is time-independent, by Liouville's theorem, the invariance of volumes in phase space under time translation. Since energy is conserved, we can insert into the measure the a time-independent delta function of energy

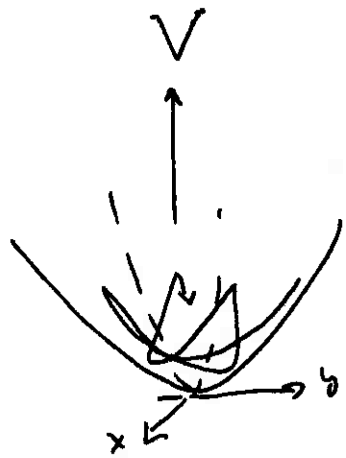
$$\int dq\, dp \;\; \delta(E - H(q,p))$$

For an ergodic system, this would be the only possible time-independent measure on phase space.

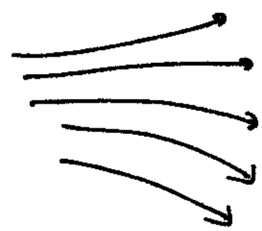
The basic principle of statistical mechanics is to use this measure on phase space to describe thermal equilibrium states. You will see in your statistical mechanics course that this hypothesis gives detailed predictions in agreement with experiment for gases, fluids, and solids, and for many more general mechanical systems as well. In this sense, then, it would be a good thing if it turned out that phase space trajectories were as complex as possible.

The ideas of integrability and ergodicity are two extremes for the behavior of a Hamiltonian system. On one hand, there are systems with complex nonlinear dynamics that are known to be integrable. On the other hand, there are some systems that are known to be ergodic. For example, Sinai provide ergodicity for the mechanics of hard spheres in 3 dimensions. It is also possible for a system to have dynamics that is some intermediate case between these two extremes. We will see examples of this in the next two weeks. As to the question of what is the typical behavior of a Hamiltonian system, this is still a matter for research with no clear answer.

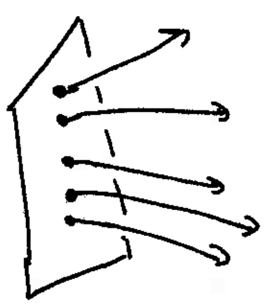
How can a system be partway between integrability and ergodicity. To explain this, I will first introduce a very useful tool for the analysis of Hamiltonian systems, the *Poincaré section.* Imagine a mechanical system with n degrees of freedom in which the size of orbits is bounded, for example, motion in a potential well in n dimensions.

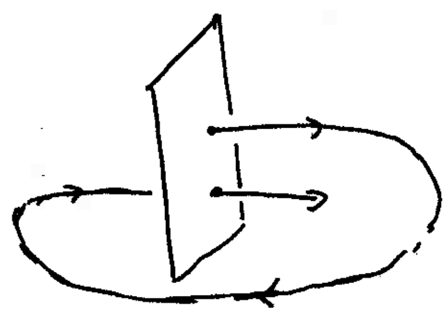


Poincaré introduced a method to visualize the orbits in such a system. First, fix the energy. Then we have flows in a space of $(2n-1)$ dimensions. We might visualize the flows as laminating the space,



Now construct a surface that is transverse to the flow, that is, a surface such that no trajectory is tangent to the surface. The intersection of a trajectory with the surface will be a point. This surface is the *Poincaré section.* If the particle motions form bounded orbits, then eventually a trajectory will return and intersect the Poincaré section again.



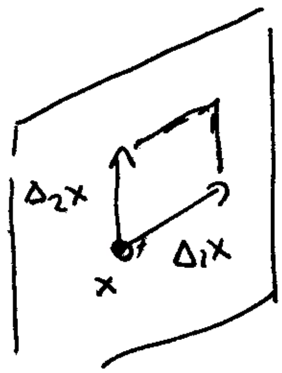


Poincaré tried to understand the 3-body problem of celestial mechanics by studying periodic trajectories that would intersect the Poincaré section repeatedly in the same point. This is the simplest possible behavior of a trajectory with respect to the section.

The Poincaré section has a remarkable property inherited from the invariance of the area in phase space under a Hamiltonian flow. If we consider the area of a 2-dimensional region on the Poincaré section, evolve the points in this region forward in time until they intersect the section again, and then measure the area of the resulting region, those two areas will be the same. The shape of the area might be distorted by a rotation or a compression and expansion along orthogonal axes, but the principle of invariance of the area in phase space holds for this mapping. I will now prove this statement.

To begin, construct a small parallelogram $\mathcal{P}$ on the Poincaré section. Call the base point of the parallelogram

$$x = (q, p)$$



and the sides of the parallelogram

$$\Delta_1 x = (\Delta_1 q, \Delta_1 p) \qquad \Delta_2 x = (\Delta_2 q, \Delta_2 p)$$

The area of the parellelogram $\mathcal{P}$ is

$$a = \Delta_1 q_i \, \Delta_2 p_i - \Delta_1 p_i \, \Delta_2 q_i = \Delta_1 x_i \, E_{ij} \, \Delta_2 x_j$$

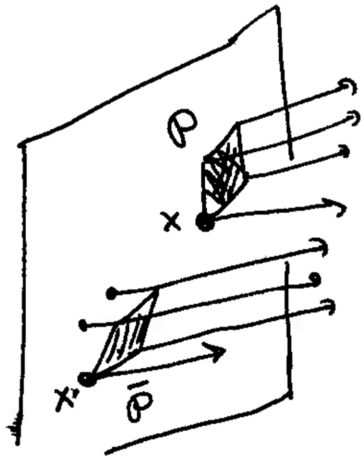
I will write this as

$$a = \Omega(\Delta_1 x, \Delta_2 x)$$

defining the symplectic-invariant product Ω of two vectors $\Delta_i x$. This area is exactly the Poincaré invariant integral over the parallelogram. To make the connection explicit, let α, β be coordinates on the parallelogram.

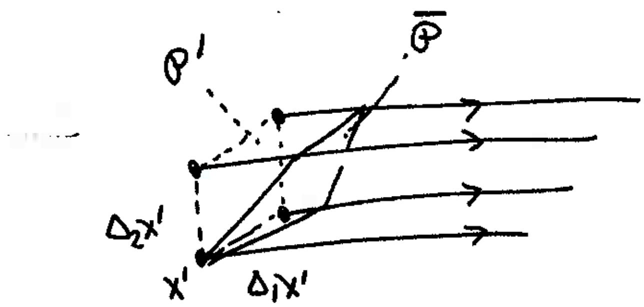
We showed in an earlier lecture that the value of the Poincaré invariant integral is unchanged by canonical transformations. So, if we carry the parallelogram $\mathcal{P}$ forward in time until the image of the point $x = (q, p)$ intersects the Poincaré section again, in a point

$$x' = (q', p')$$



then the resulting image parallelogram $\overline{\mathcal{P}}$ will have the same area as $\mathcal{P}$.

However, this is not exactly what we want. Since neighboring trajectories will not pierce the Poincaré section at the same time, the parallelogram $\overline{\mathcal{P}}$ will not lie in the section. We are interested in comparing the area of $\mathcal{P}$ to that of a parallelogram $\mathcal{P}'$ with base point x' that lies in the section. The relation between $\mathcal{P}'$ and $\overline{\mathcal{P}}$ is shown in this figure:



Let the sides of the parallelogram $\mathcal{P}'$ be

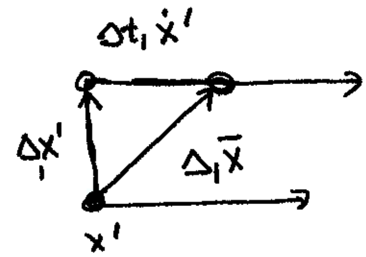
$$\Delta_1 x' = (\Delta_1 q', \Delta_1 p')$$

$$\Delta_2 x' = (\Delta_2 q', \Delta_2 p')$$

Then, if Δt_1 be the time difference between the time that the reference tranjectory from x to x' passes through the Poincaré section and the time that the trajectory 1 passes through the Poincaré section. Define Δt_2 similarly. Then the sides of the parallelogram $\overline{\mathcal{P}}$ are

$$\Delta_1 \bar{x} = \Delta_1 x' + \Delta t_1 \dot{x}'$$

$$\Delta_2 \bar{x} = \Delta_2 x' + \Delta t_2 \dot{x}'$$



The area of $\mathcal{P}'$ is

$$a' = \Omega(\Delta_1 x', \Delta_2 x')$$

The area of $\overline{\mathcal{P}}$ is

$$\bar{a} = \Omega(\Delta_1 x' + \Delta t_1 \dot{x}', \Delta_2 x' + \Delta t_2 \dot{x}')$$
$$= \Omega(\Delta x_1', \Delta_2 x') + \Delta t_2 \, \Omega(\Delta x_1', \dot{x}') + \Delta t_1 \, \Omega(\dot{x}', \Delta_2 x') + \Delta t_1 \Delta t_2 \, \Omega(\dot{x}', \dot{x}')$$

The first term is the area of $\mathcal{P}'$. The last term vanishes by the antisymmetry of $\Omega(.,.)$ The second term is written more explicitly as

$$\Omega(\Delta x_1', \dot{x}') = \Delta q_i' \, \dot{p}_i' - \Delta p_i' \, \dot{q}_i$$
$$= \Delta q_i' \left(\frac{\partial H}{\partial q_i}\right)' - \Delta p_i' \left(\frac{\partial H}{\partial p_i}\right)'$$
$$= -\left(\Delta q_i \frac{\partial H}{\partial q_i} + \Delta p_i \frac{\partial H}{\partial p_i}\right)'$$

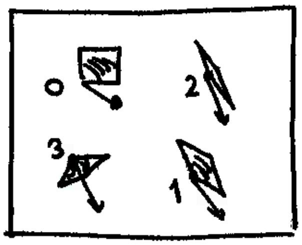
But all of the trajectories we consider are on a surface of constant H. Thus

$$\Delta q_i \frac{\partial H}{\partial q_i} + \Delta p_i \frac{\partial H}{\partial p_i} = \Delta H = 0$$

and so this term vanishes. The third term $\Omega(\Delta_1 t\dot{x}, \Delta_2 x)$ vanishes in a similar way. Then

$$a' = \bar{a} = a$$

Thus, if a bundle of trajectories pierce the Poincaré section in a 2-dimensional region of some area, the next intersection of these trajectories with the Poincaré section will be a 2-dimensional region with the same area. The region may of course be distorted. It might be rotated, or stretched along one axis and compressed along the other

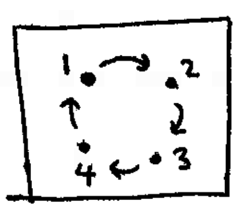


This result is a part of a stronger theorem proved by Darboux: Locally in phase space, there exists a canonical transformation to new coordinates that include the Hamiltonian as one of the momenta, a appropriately chosen coordinate running along the trajectories, and $(n-1)$ pairs of coordinates and momenta lying in the Poincaré section.

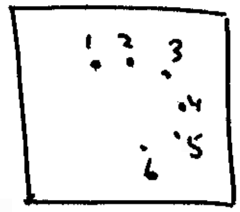
It is easy to picture the intersections of a trajectory with the Poincaré section in the case of an integrable system. I will consider the case of a system with 2 degrees of freedom for clarity. First, transform to action-angle variables $(\phi_1, J_1, \phi_2, J_2)$. The motion is then translation on a torus, with the motion in each coordinate periodic with the frequencies

$$\omega_1 = \frac{\partial H}{\partial J_1} \qquad \omega_2 = \frac{\partial H}{\partial J_2}$$

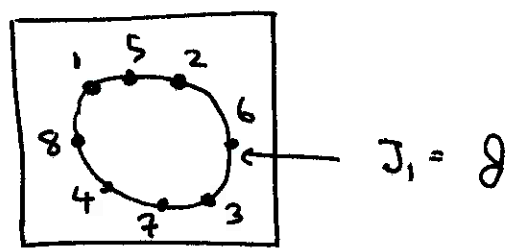
If $\omega_1 = \omega_2$, the motion is periodic, the trajectory pierces the Poincaré section once. If ω_1 and ω_2 are *commensurate*, $\omega_1/\omega_2 = n/m$ for integers n, m, then the motion is periodic but the trajectory will intersect the section multiple times before returning to the original point.



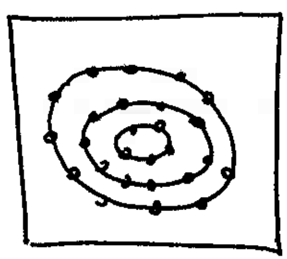
More generally, ω_1 and ω_2 are incommensurate. Then the motion is aperiodic and the trajectory will intersect the Poincaré section in a new point on every orbit.



However, if the system is integrable, J_1 and J_2 will be separately conserved. For fixed H, J_1 is an independent variable. If we draw the curve of fixed J_1 on the Poincaré section, all of the intersections of an orbit must lie on that curve.



The section will then appear as a set of nested curves corresponding to the different possible values of J_1,

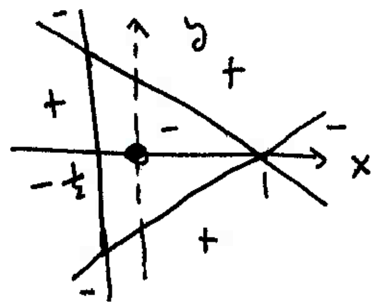


An integrable system with more degrees of freedom will give a similar set of pictures in a higher dimensional space. Note that a system with one degree of freedom is too simple, since the phase space trajectories are completely fixed by the conservation of H.

It is interesting, then, to examine model systems with two degrees of freedom and see if they correspond to the expectation for integrable systems. One of the earliest systems that was studied systematically was the motion of a particle in a two-dimensional potential constructed by Hénon and Heiles. Hénon and Heiles were astronomers studying the motion of stars in galaxies. As a model potential with a deep minimum near the origin and the possibility of escape to infinity, they wrote

$$U(x,y) = \left(y+\tfrac{1}{2}\right)\left(x-\frac{y-1}{\sqrt{3}}\right)\left(x+\frac{y-1}{\sqrt{3}}\right)+\frac{1}{6}$$

The first term of the potential is zero on an equilateral triangle



The potential has a local minimum with $U = 0$ at $(x, y) = 0$. On the equilateral triangle, $U = \frac{1}{6}$. Beyond the triangle, U goes up or down from this value as indicated in the figure.

Hénon and Heiles studied the Hamiltonian

$$H = \frac{p_x^2 + p_y^2}{2} + U(x, y)$$

looking, in particular, at the Poincaré section given by setting $x = 0$ on a slice of phase space of constant energy. The section can be parametrized by y and p_y, with the allowed region satisfying

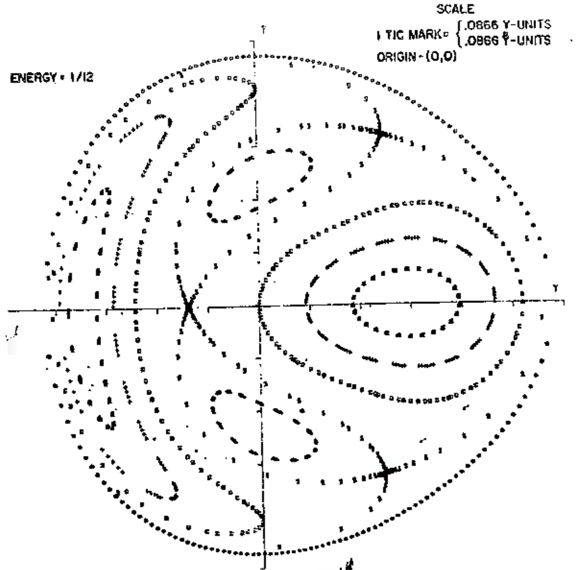
$$\frac{p_y^2}{2} + U(x, y) < E$$

The successive intersections of some trajectories for the values of the energy

$$E = \frac{1}{12}, \frac{1}{8}, \frac{1}{6}$$

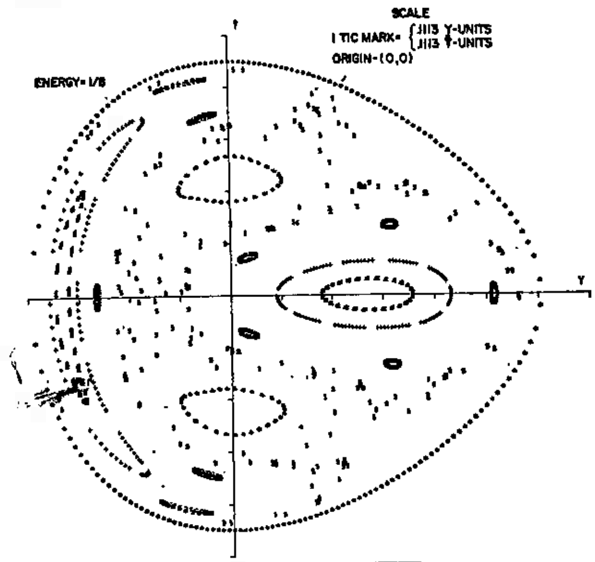
are shown in the figure (taken from F. Gustavson, Astron. J. 71, 670 (1966)). At $E = 1/12$, the points of intersection seem to lie on smooth curves in the (y, p_y) plane. For $E = 1/8$, however, these smooth curves seem to be restricted so localized regions of the plane. The points in the plane away from these regions are actually all generated by a single trajectory that eventually fills this intermediate area densely. The regions with

$E = \frac{1}{12}$



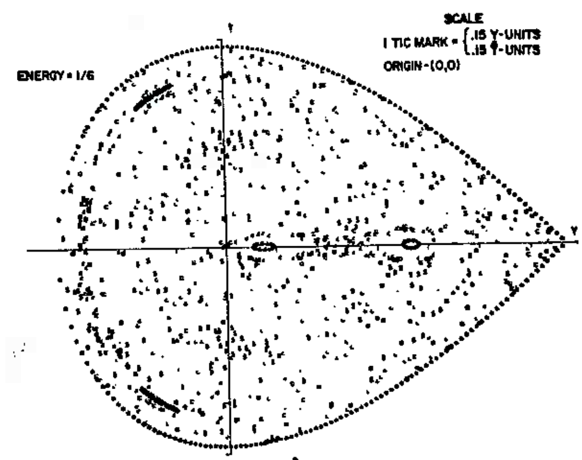
STARTING POINT	I_{END}	E_{END}	NO. PUNCTURES
× – (.16,0)	618	.0833217	200
+ – (.08,0)	627	.0833217	200
□ – (0,0)	640	.0833220	200
$ – (-.12,0)	666	.0833219	200
◇ – (0,.16)	686	.0833216	200
* – ZERO VELOCITY CURVE			

$E = \frac{1}{8}$



STARTING POINT	I_{END}	E_{END}	NO. PUNCTURES
× – (.2,0)	611	.124983	200
+ – (.1,0)	622	.124984	200
⊡ – (.02,.08)	802	.124979	250
$ – (-.17568,0)	674	.124983	200
◇ – (0,.2)	743	.124982	200
* – ZERO VELOCITY CURVE			

$E = \frac{1}{6}$



STARTING POINT	I_{END}	E_{END}	NO. PUNCTURES
× – (.3,0)	623	.1666444	200
+ – (.08,0)	635	.1666437	200
□ – (0,0)	614	.1666468	200
$ – (-.222,0)	763	.1666438	200
◇ – (0,.25)	688	.1666451	200
* ZERO VELOCITY CURVE			

intersections on smooth curves are called *stable islands.* As the energy is increased further, these stable islands shrink in size and eventually disappear. The situation at $E = 1/6$ shows only a few small islands remaining. Oddly, though the Hénon-Heiles system is not integrable, it is not ergodic either, though it becomes closer to ergodic as the energy increases.

The behavior in which a single trajectory pierces the Poincaré section in a seemingly unpredictable way and eventually fills the phase space is one that we could reasonably call *chaotic.* Chaos is an imperfectly understood concept and still lacks a completely general theory. However, there are many examples of system that are, intuitively, chaotic and share certain tangible properties. We can use these properties to characterize chaos in a physical system.

Here are three properties shared by all chaotic systems:

1. *Sensitivity to initial conditions*: A chaotic system is unpredictable, in the sense that, when the equations of mechanics are integrated with very similar initial conditions, the solutions quickly diverge from one another. Nearby initial conditions lead to trajectories that end up, at a given time, in very different regions of phase space. This divergence can be characterized in the following way: Choose an initial condition x_1 and another nearby point x_2 such that

$$|\vec{x}_1 - \vec{x}_2| = \epsilon$$

Then the distance between the trajectories that emerge from these points grows exponentially

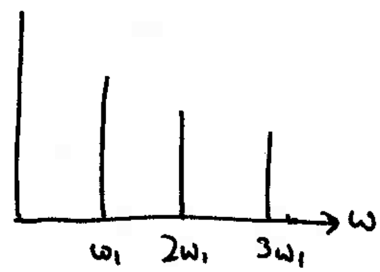
$$|\vec{x}_1(t) - \vec{x}_2(t)| \sim \epsilon\, e^{\lambda t}$$

The parameter λ is called the *Lyapunov exponent.* In most examples the distance $|x_1(t) - x_2(t)|$ is eventually bounded, but the two trajectories still occupy different positions, at apparently random distances, within the phase space.

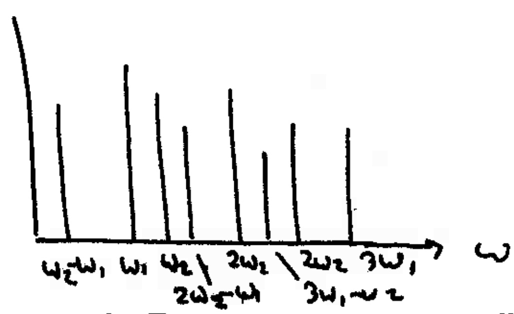
2. *Space-filling trajectories*: A chaotic system will have trajectories that are dense in a subregion of the phase space. That is, a single trajectory will come arbitrarily close to all points in a region of the phase space of dimensionality higher that 1. In an ergodic system, a trajectory is dense in the full phase space, but

not all examples have this property. More typically, the trajectory is a *fractal*, a curve of fractional dimension, that fills part of the phase space.

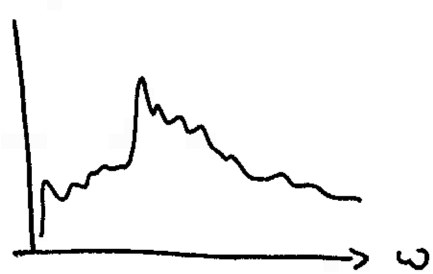
3. *Existence of periodic trajectories with all possible frequencies.* A chaotic system will have periodic trajectories that are unstable with respect to the space-filling trajectories. However, these periodic trajectories can be used as a tool to understand the chaotic dynamics. As an example, I will describe a method that is often used to observe the transition to chaos experimentally. At some values of a parameter, before the motion becomes chaotic, a system might be characterized by periodicity at a definite frequency ω_1. The Fourier spectrum of this motion will be a set of spikes,



At a higher value of the parameter, a second frequency ω_2 might appear. The spectrum still has Fourier components only at definite values, for example,



For a chaotic system, the Fourier spectrum covers all frequencies.



I will discuss later some ways that the spectrum with discrete Fourier components can be converted to one with power at all frequencies. It should be clear that careful experimentation is needed to recognize chaos. Every experimental spectrum has noise, and no peaks of the Fourier spectrum are perfect delta-functions. However, it is possible to observe clearly transitions between these two types of behavior.

Chaotic behavior and transitions from regular to chaotic motion can be studied at three different levels. At the highest level, we could study Hamiltonian systems, using the full apparatus of Hamiltonian mechanics. Relaxing some constraints, we could study simple systems of ordinary differential equations. Finally, visualizing the motion of a mechanical system as a set of points on the Poincaré section, we could study discrete recursions. In the rest of this lecture and in the next lectures, I will give examples of chaotic systems at all three of these levels. Many of the original papers on these examples are collected in the reprint volume *Universality in Chaos*, edited by Predrag Cvitanovic (Adam Hilger, 1984).

The simplest place to start is actually with ordinary differential equations. An example of a system of differential equations with chaos is the Lorenz model, introduced by Edward Lorenz in 1963. Lorenz was studying convection in the atmosphere and was trying to find a simple set of differential equations that modelled with instability to convection. We have already seen that a set of 2 differential equations is too simple, because the constraint that trajectories cannot cross in the phase plane is too strong a constraint on the possible motions. Lorenz wrote a set of 3 differential equations:

$$\dot{x} = -\sigma x + \sigma y$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz$$

The behavior of these equations does not depend strongly on the values of σ and b. Lorenz, in his original paper, chose the values $\sigma = 10$, $b = 8/3$ for ease of numerical integration. The variable r is a *control parameter.* We will see the behavior change qualitatively as r is increased from $r = 0$ to large values.

To begin our analysis of this model, notice that there is a stationary solution at

$$x = y = z = 0$$

It is easy to work out the stability of this point. The linearized equations are

$$\dot{x} = -\sigma x + \sigma y$$
$$\dot{y} = rx - y$$
$$\dot{z} = -bz$$

forming the matrix system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & \\ r & -1 & \\ & & -b \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The z direction is obviously stable; a deviation z decays as

$$z(t) = z(0)\, e^{-bt}$$

For the remaining two variables, the characteristic equation is

$$\lambda^2 + (\sigma+1)\lambda + \sigma(1-r) = 0$$

For $r = 0$, this equation is

$$(\lambda+\sigma)(\lambda+1) = 0$$

giving the eigenvalues

$$\lambda = -1, -\sigma$$

For larger r, the product of eigenvalues is

$$\lambda_1 \lambda_2 = \sigma(1-r)$$

This is positive for $r < 1$, and, indeed, both eigenvalues remain negative in that region. At $r = 1$, one eigenvalue goes through zero and becomes positive; then there is an instability for $r > 1$.

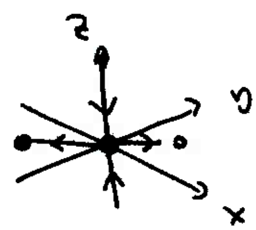
There are two additional stationary solutions to the Lorenz system. Putting $\dot{x} = \dot{y} = \dot{z} = 0$, we find the equations

$$x = y$$
$$xy = bz$$
$$rx - y = xz$$

Putting $y = x$ into the second and third equations, we find the previous fixed point and two new ones,

$$x = y = \pm\sqrt{b(r-1)} \qquad z = r-1$$

For $r > 1$, the fixed point at $x = y = z = 0$ is unstable, with flows going outward to the other two



We should now analyze the stability of the new fixed points. I will analyze the point $x = y = +\sqrt{b(r-1)}$; the analysis for the other point is similar. The linearized equations take the matrix form

$$\begin{pmatrix} \dot{x} \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -x_* \\ x_* & x_* & -b \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad x_* = \sqrt{b(r-1)}$$

The eigenvalues are determined by the characteristic equation

$$\lambda^3 + (\sigma+b+1)\lambda^2 + (r+\sigma)b\lambda + 2\sigma b(r-1) = 0$$

For r very close to 1, there is a small eigenvalue that we can determine by balancing the λ^1 and constant terms in this equation.

$$\lambda \cong -\frac{2\sigma}{1+\sigma}(r-1)$$

The remaining eigenvalues satisfy

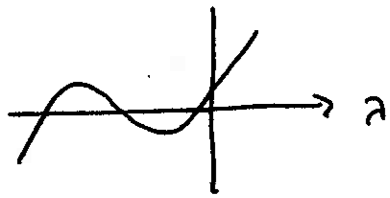
$$\lambda^2 + (\sigma+b+1)\lambda + (1+\sigma)b = 0$$

and thus have the values

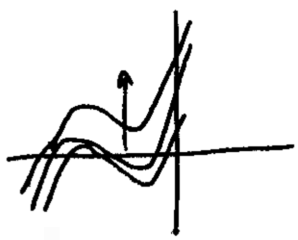
$$\lambda \cong -(\sigma+1), -b$$

All three eigenvalues are negative, so the point is completely stable.

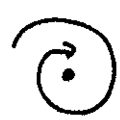
The left-hand side of the characteristic equation for small values of $(r-1)$ is a cubic polynomial of the form



As r is increased, the polynomial deforms to



Eventually, the minimum of the cubic lifts up above the real axis. Then we have one negative and a pair of complex solutions. These complex solutions, however, have a negative real part and give orbits that spiral in to the fixed point



At some point, however, the real part of these eigenvalues goes through zero and becomes positive. A cubic with one negative and two pure imaginary roots has the form

$$(\lambda+\alpha)(\lambda+i\beta)(\lambda-i\beta) = 0$$

or

$$\lambda^3 + \alpha\lambda^2 + \beta^2\lambda + \alpha\beta^2 = 0$$

The characteristic equation above is of this form when

$$(\sigma+b+1)(r+\sigma) = 2\sigma b(r-1)$$

This is an equation that can be solved for r. The solution gives the point at which the new fixed points become unstable

$$r = \frac{\sigma(\sigma+b+3)}{(\sigma-b-1)} \approx 24.7 \text{ for } \sigma=10 \quad b=8/3$$

If $\sigma < (b+1)$, the fixed points are always stable, but if $\sigma > (b+1)$ and r is greater than the value given by this equation, the trajectories will spiral out of the new fixed points. Where do they go?

Before answering this question, I will give another property of the phase flows for the Lorenz system. The transformation from (x, y, z) to the time-evolved point $(x(t), y(t), z(t))$ has a Jacobian

$$\mathcal{J}(t) = \frac{\partial(x(t)\ y(t)\ z(t))}{\partial(x, y, z)}$$

For small times, the Jacobian is

$$\mathcal{J}(\Delta t) = 1 + \Delta t\ \frac{\partial(\dot{x}\ \dot{y}\ \dot{z})}{\partial(x, y, z)}$$

We can evaluate the determinant of this expression by relating it to the trace of the deviation from 1, as we did in the proof of Liouville's theorem. We find

$$\det \mathcal{J}(\Delta t) = 1 + \Delta t\ \mathrm{tr}\ \frac{\partial(\dot{x}\ \dot{y}\ \dot{z})}{\partial(x\ y\ z)}$$

$$\det \mathcal{J}(\Delta t) = 1 + \Delta t \; \mathrm{tr} \begin{bmatrix} -\sigma & & \\ & -1 & \\ & & -b \end{bmatrix}$$

$$= 1 - \Delta t\,(\sigma + b + 1)$$

The changes in $\det \mathcal{J}$ accumulate, and thus the determinant of the Jacobian obeys the equation

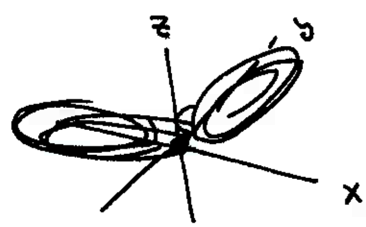
$$\frac{d}{dt}\left[\det \mathcal{J}(t)\right] = \left[\det \mathcal{J}(t)\right] \cdot \left[-(\sigma + b + 1)\right]$$

whose solution is

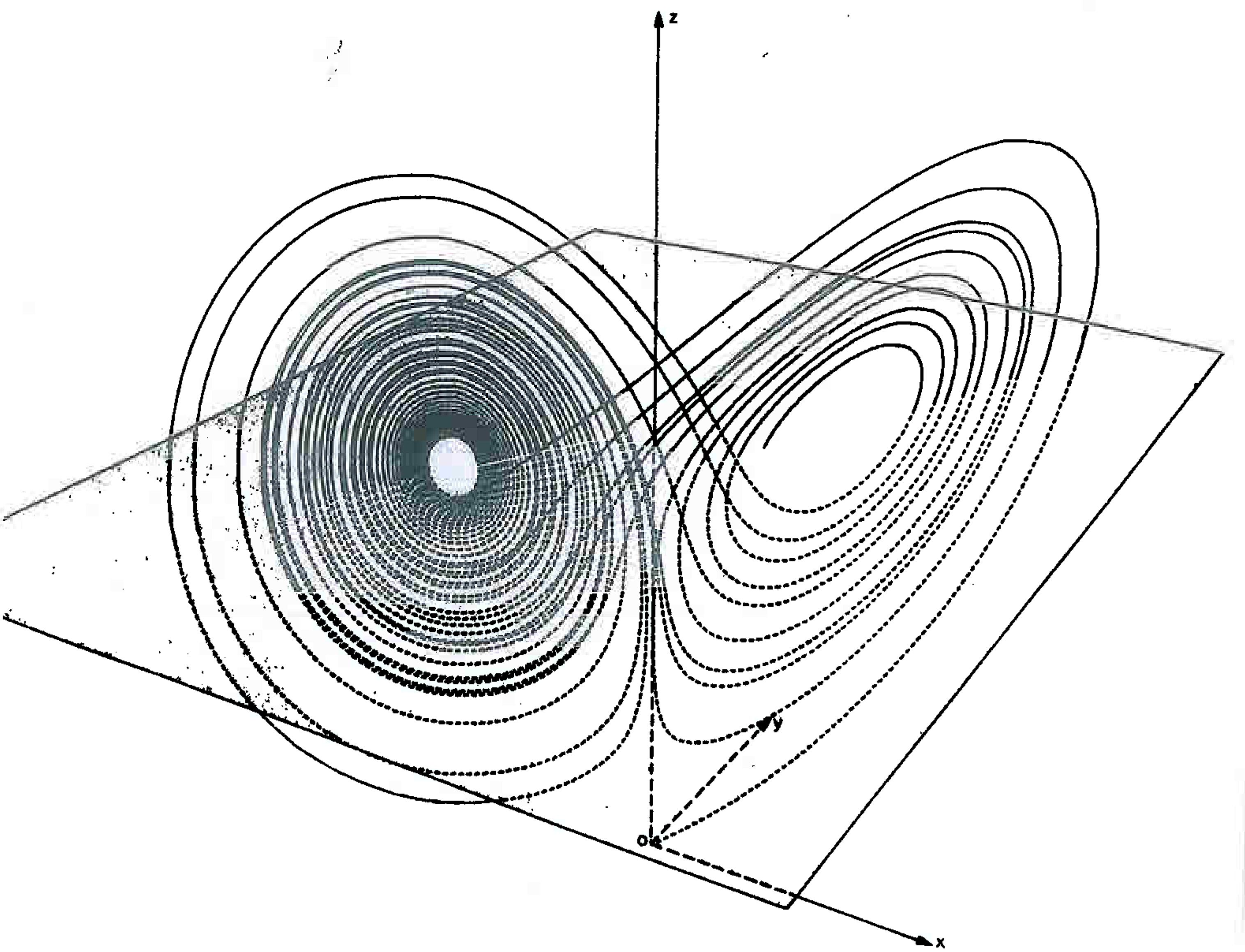
$$\det \mathcal{J}(t) = e^{-(\sigma + b + 1)t}$$

Thus, under the dynamics of the Lorenz system, the volume in phase space continually decreases.

However, in the regime where the fixed points are unstable, it turns out that the amplitude of the motions does not decrease. Instead, the trajectories are attracted to a pair of 2-dimensional regions



A more careful drawing of the attractor is shown in the figure, taken from D. Ruelle, Math. Intelligencer 2, 126 (1980). This attractor is more complex than the attractive points and limit cycles that we have studied in the course up to now. It is actually a fractal with dimensionality close to 2. It is technically characterized as a *strange*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attractor. The motion in this attractor is chaotic, satisfying all of the three criteria listed above. The first two properties have already been discribed. Lorenz, in his original paper, constructed periodic solutions in the attractor with a variety of periods.

If chaotic behavior can arise in such a simple system, it must arise in many other places. I will give more examples in the next two lectures.