Sept. 23

Central Forces

Before we begin our discussion of advanced methods in mechanics, I will take one lecture to derive some additional important results that can be obtained from Newton's laws. These concern particles interacting through central forces, that is, force laws of the form

$$\vec{F}_{i\leftarrow j} \propto (\overrightarrow{r_i - r_j})$$

I will begin with standard facts about central force motion, and then consider some related identities that are less well known.

Consider first a single particle in motion around a center of force. For a central force, Newton's law for this system is

$$m\ddot{\vec{r}} = \vec{F}(\vec{r}) = -\hat{r}\, F(r)$$

This illustrates the notation that I will adopt for the rest of the course: $\vec{r}$ is a vector with components r^i, $r = |\vec{r}|$ is the magnitude of $\vec{r}$, and $\hat{r}$ is a unit vector, $|\hat{r}| = 1$, in the direction of $\vec{r}$.

This system has two well known integrals of the motion. The first is the *energy*

$$E = \frac{1}{2} m (\dot{\vec{r}})^2 + V(r)$$

where the *potential energy* $V(r)$ is given by

$$V(r) = \int_{r_0}^{r} dr'\, F(r')$$

For a potential that falls off fast enough as $r \to \infty$, we choose the constant of integration so the $V(r) \to 0$ as $r \to \infty$,

$$V(r) = - \int_r^{\infty} dr' F(r')$$

Then we can check explicitly,

$$\frac{dE}{dt} = m \dot{\vec{r}} \cdot \ddot{\vec{r}} + \dot{r} \frac{\partial}{\partial r} V$$

$$= \dot{\vec{r}} \cdot \left[m \ddot{\vec{r}} + \hat{r} F(r) \right] = 0$$

The second integral of the motion is the *angular momentum*

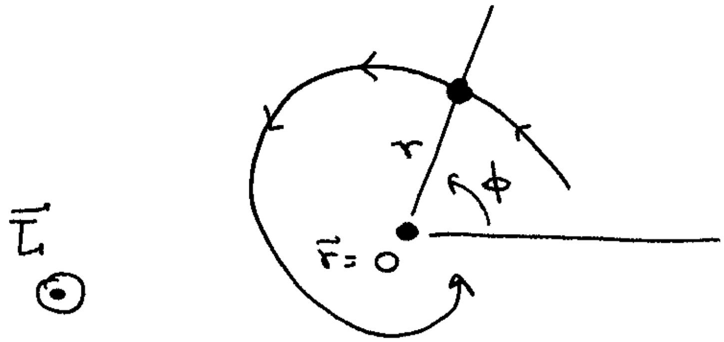
$$\vec{L} = m \vec{r} \times \dot{\vec{r}}$$

Explicitly,

$$\frac{d\vec{L}}{dt} = m \dot{\vec{r}} \times \dot{\vec{r}} + m \vec{r} \times \ddot{\vec{r}}$$

$$= 0 + m \vec{r} \times \left(- \frac{1}{m} \hat{r} F(r) \right) = 0$$

Motion in a central potential in 3 dimensions requires 6 initial conditions, the values of $\vec{r}$ and $\dot{\vec{r}}$ at $t = 0$. Four of these variables can be exchanged for the values of E and $\vec{L}$, which are constant. This leaves two variables that change with time, which we can describe in the following way: By the definition of $\vec{L}$, $\vec{L}$ is perpendicular to both $\vec{r}$ and $\dot{\vec{r}}$. Since $\vec{L}$ is constant, the motion then occurs in the plane through $\vec{r} = 0$ that is orthogonal to the fixed vector $\vec{L}$. The orbit in that plane has the form



We can parametrize the particle motion by cylindrical coordinates in this plane $(r(t), \phi(t))$. In terms of (r, ϕ), the scalar conserved quantities are given by

$$|\vec{L}| = m r^2 \dot{\phi}$$

$$E = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}^2 \right) + V(r)$$

so the four initial conditions $(r, \dot{r}, \phi, \dot{\phi})$ are restricted by the choice of E, $\vec{L}$ to two free conditions.

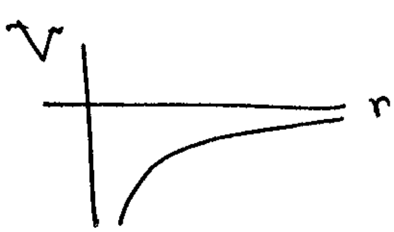
In cylindrical coordinates, E takes the form

$$E = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} \frac{\vec{L}^2}{m r^2} + V(r)$$

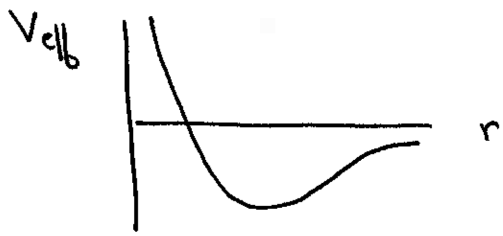
For the concrete case of gravitational attraction,

$$F = \frac{GMm}{r^2} \qquad V(r) = \frac{GMm}{r}$$

Then a particle with zero angular momentum sees the potential energy



while a particle with $L \neq 0$ sees in addition an *angular momentum barrier*

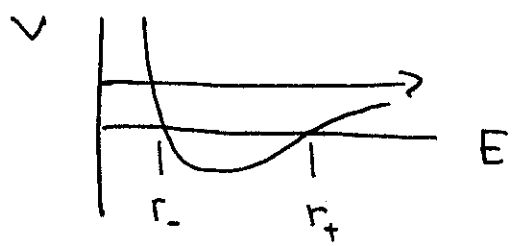


$$V_{eff}(r) = \frac{L^2}{2mr^2} + \frac{GMm}{r}$$

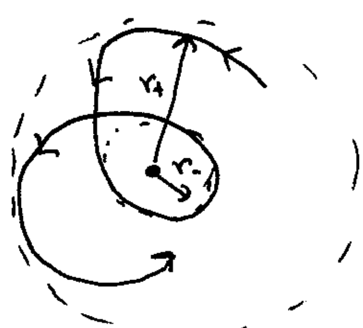
For any force law of the general form that it falls to zero as $r \to \infty$ such that

$$F(r) < \frac{A}{r^3} \quad \text{as } r \to 0 \qquad F(r) < \frac{B}{r} \quad \text{as } r \to \infty$$

for any A, B, a particle with nonzero angular momentum and $E < 0$ will be trapped in this effective potential in a finite region



Pictorially,



The particle must orbit between the radii r_- and r_+. The case of $F(r) = C/r^2$ is special; I will discuss this in more detail later.

Next, consider a system of N particles interacting with one another through central forces. The particle i obeys the Newton equation

$$m_i \ddot{\vec{r}}_i = \sum_{j \neq i} \vec{F}_{i \leftarrow j}$$

where the force exerted by particle j on the particle i is

$$\vec{F}_{i \leftarrow j} = - \hat{r}_{ij} \, F(r_{ij})$$

with

$$\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$$

In my conventions

$$\hat{r}_{ij} = \frac{\vec{r}_i - \vec{r}_j}{|\vec{r}_i - \vec{r}_j|} \qquad\qquad r_{ij} = |\vec{r}_i - \vec{r}_j|$$

Like all isolated systems interacting through conservative forces, this system has a conserved energy, momentum, and angular momentum. It is instructive to work out the full set of conservation laws explicitly.

Begin with the energy. Define the kinetic energy T and the potential energy V by

$$T = \sum_i \tfrac{1}{2} m_i |\dot{\vec{r}}_i|^2 \qquad\qquad V = \sum_{i<j} V(r_{ij})$$

where

$$V(r) = -\int_r^\infty dr' F(r')$$

as before. This energy is conserved: $dE/dt = 0$. Here is the proof:

$$\begin{aligned}\frac{dE}{dt} &= \sum_i m_i \dot{\vec{r}}_i \ddot{\vec{r}}_i + \sum_i \sum_{j \neq i} \dot{\vec{r}}_i \cdot \frac{\partial}{\partial \vec{r}_i} V(r_{ij}) \\ &= \sum_i \dot{\vec{r}}_i \cdot \left(m \ddot{\vec{r}}_i + \frac{\partial}{\partial \vec{r}_i} \sum_{j \neq i} V(r_{ij}) \right) \\ &= \sum_i \dot{\vec{r}}_i \cdot \left(m \ddot{\vec{r}}_i + \sum_{j \neq i} \hat{r}_{ij} F(r_{ij}) \right) = 0\end{aligned}$$

The last line vanishes term by term using the equations of motion.

The angular momentum is

$$\vec{L} = \sum_i m_i \vec{r}_i \times \dot{\vec{r}}_i$$

The conservation of $\vec{L}$, $d\vec{L}/dt = 0$, is proved as follows:

$$\begin{aligned}\frac{d\vec{L}}{dt} &= \sum_i \left(m_i \dot{\vec{r}}_i \times \dot{\vec{r}}_i + m_i \vec{r}_i \times \ddot{\vec{r}}_i \right) \\ &= \sum_i \left(0 + \sum_{j \neq i} \vec{r}_i \times \left(-\hat{r}_{ij} F(r_{ij}) \right) \right)\end{aligned}$$

This expression can be organized into a sum over pairs (i, j)

$$\begin{aligned}&= \sum_{(ij)} \left(\vec{r}_i \times \hat{r}_{ij} + \vec{r}_j \times \hat{r}_{ji} \right) F(r_{ij}) \\ &= \sum_{(ij)} \left(\vec{r}_i - \vec{r}_j \right) \times \hat{r}_{ij} F(r_{ij}) = 0\end{aligned}$$

This vanishes because $(\vec{r}_i - \vec{r}_j)$ is parallel to $\hat{r}_{ij}$.

The total momentum is

$$\vec{P} = \sum_i m_i \dot{\vec{r}}_i$$

Conservation of momentum, $d\vec{P}/dt = 0$, follows from

$$\frac{d\vec{P}}{dt} = \sum_i m_i \ddot{\vec{r}}_i = \sum_i \Big(- \sum_{j \neq i} \hat{r}_{ij} F(r_{ij}) \Big)$$

$$= - \sum_{(ij)} (\hat{r}_{ij} + \hat{r}_{ji}) F(r_{ij}) = 0$$

that is, from the fact that central forces satisfy Newton's law of equal action and reaction.

Notice that $\vec{P}$ is itself a total derivative

$$\vec{P} = \frac{d}{dt}\Big(\sum_i m_i \vec{r}_i \Big) = M \dot{\vec{R}}$$

where

$$M = \sum_i m_i$$

is the total mass of the particles, which is fixed, and

$$\vec{R} = \sum_i \frac{m_i}{M} \vec{r}_i$$

is the center of mass of the particles. Conservation of momentum implies constant motion of the center of mass

$$\vec{R} = \frac{\vec{P}}{M}(t - t_0)$$

It is usually convenient to make a (Galilean) boost to the frame where $\vec{P} = 0$, the center of mass frame. In this frame, the center of mass is at rest.

The Newton equations of a system of N particles in 3 dimensions require $6N$ initial conditions. Of these, 6 are fixed by the specification of $\vec{P}$ and $\vec{R}$, and 4 more are fixed by the specification of E and $\vec{L}$. The nontrivial interactions of the particles involves motion in the space perpendicular to these fixed directions, that is, in a phase space of $6N - 10$ coordinates.

In the case of two particles, the space not fixed by conservation laws has 2 dimensions. Essentially, it is the same 2 dimensional space discussed above for the motion of a particle around a center of force. It is useful, for a 2-particle system, to explicitly eliminate $\vec{P}$, $\vec{R}$ and reduce the problem to one involving one relative vector $\vec{r}$. This reduction of a 2-particle system to what is effectively a 1-particle system comes up in many physical systems. I will give the formulae here; it is worth memorizing them.

Let

$$\vec{R} = \frac{m_1}{M}\vec{r}_1 + \frac{m_2}{M}\vec{r}_2 \qquad \vec{r} = \vec{r}_1 - \vec{r}_2$$

$$M = m_1 + m_2$$

We can compute $\ddot{\vec{r}}$ from Newton's laws,

$$\ddot{\vec{r}} = -\frac{1}{m_1}\hat{r}_{12}F(r_{12}) + \frac{1}{m_2}\hat{r}_{21}F(r_{12})$$

But

$$\hat{r}_{12} = -\hat{r}_{21} = \hat{r} \qquad r_{12} = r$$

so the two terms are identical in form and can be combined into

$$\ddot{\vec{r}} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right) \hat{r}\, F(r)$$

This is canoncially written

$$\mu \ddot{\vec{r}} = - \hat{r}\, F(r)$$

where

$$\frac{1}{\mu} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right) \qquad \text{or} \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

The quantity μ is called the *reduced mass.* For $m_2/m_1 \to \infty$, $\mu \to m_1$, as one might expect. But, when $m_2 = m_1$, we find rather counterintuitively that the 2-particle system is equivalent to a central force problem with the mass

$$\mu = \frac{1}{2} m_1 \qquad\qquad (m_1 = m_2)$$

This factor of 2 gives a factor of 2 decrease in the binding energy, which is observed in many systems from positronium to binary stars.

The reduced mass played a role in Bohr's introduction of the Bohr quantum theory of the atom. In 1896, Charles Pickering had identified spectral lines in starlight that did not fit the Balmer formula. Bohr pointed out that these lines fit the Balmer formula for He^+, with the states of single-electron Helium following from his model. Alfred Fowler objected that, to fit the data, that ratio energy spacings in the two Balmer series should be 4.0016, rather than $4 = Z^2$ in Bohr's theory, and that the difference was outside the experimental error. Bohr replied that, when one included the effect of the reduced mass in Hydrogen and Helium, the result would be

$$Z^2 \left(1 + \frac{m_e}{m_{He}}\right)^{-1} \left(1 + \frac{m_e}{m_H}\right) = 4.00163$$

Touché! (See *Niels Bohr's Times*, by A. Pais, Chapter 8.)

There is one more general property of motion under central forces that applies to system which are stable over long periods of time. Let $I(t)$ be defined by

$$I = \sum_i m_i r_i^2$$

This quantity is not conserved. However, for an isolated, bound system of particles, this quantity should remain bounded.

Let me state this more precisely. For a property $Q(t)$ of a system of particles, define the time average of this property by

$$\overline{Q} = \frac{1}{T} \int_0^T dt\, Q(t)$$

and consider the limit $T \to \infty$. For an isolated, stable bound system, the limits $T \to \infty$ of the average energy, kinetic energy, and potential energy

$$\overline{E}, \overline{T}, \overline{V}$$

should all exist and be bounded. Also, the average of $I(t)$ should exist and should be bounded.

If the average of $I(t)$ over a long time is bounded, the average of $\dot{I}$,

$$\overline{\dot{I}} = \frac{1}{T} \int_0^T dt\, \dot{I}(t) = \frac{I(T) - I(0)}{T}$$

must tend to zero. Similarly,

$$\lim_{T\to\infty} \overline{\ddot{I}} \to 0$$

must tend to zero. We can study the consequences of this statement by working out the second time derivative of $I(t)$ explicitly,

$$\frac{d^2}{dt^2}\left(\sum_i m_i r_i^2\right) = \frac{d}{dt}\left(\sum_i 2m_i \vec{r}_i \cdot \dot{\vec{r}}_i\right)$$

$$= \sum_i 2m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i + \sum_i 2m_i \vec{r}_i \cdot \ddot{\vec{r}}_i$$

$$= 4\,T + \sum_i 2\vec{r}_i \cdot \left(-\sum_{j\neq i} \hat{r}_{ij} F(r_{ij})\right)$$

I have recognized the first term as the kinetic energy. The second term can be manipulated as follows:

$$2^{\text{nd}}\text{ term} = -2\sum_{(ij)} \left(\vec{r}_i \cdot \hat{r}_{ij} + \vec{r}_j \cdot \hat{r}_{ji}\right) F(r_{ij})$$

$$= -2\sum_{(ij)} (\vec{r}_i - \vec{r}_j) \cdot \hat{r}_{ij}\, F(r_{ij})$$

$$= -2\sum_{(ij)} r_{ij}\, F(r_{ij})$$

Taking the time average,

$$\overline{\ddot{I}} = 4\,\overline{T} - 2\sum_{(ij)} \overline{r_{ij}\, F(r_{ij})}$$

Setting this time average to zero, we find

$$\overline{T} = \frac{1}{2}\,\overline{\sum_{(ij)} r_{ij}\, F(r_{ij})}$$

the *virial theorem.* This is a relation between the averages over the internal motion of the kinetic and the potential energy. For example, for a force law

$$\vec{F}(r) = \frac{A}{r^{n+1}}$$

then the associated potential energy is

$$V(r) = -\frac{1}{n}\frac{A}{r^n} = -\frac{1}{n}\, r\, F(r)$$

and the virial theorem reads

$$\overline{T} = -\frac{n}{2}\,\overline{V}$$

Our proof of the virial theorem applies only to isolated systems. However, it is also useful in another case, that of a gas of particles in a box. For particles confined to a box, we must add in the forces applied to the particles by the walls of the box. However, since particles bounce back and forth between the walls, the force $\vec{f}_i$ of the wall on particle i naturally satisfies

$$\overline{\sum_i \vec{r}_i \cdot \vec{f}_i} = 0$$

Then this term cancels out of the averaged equation for $\ddot{I}(t)$ and the virial theorem is correct for this system.

For an inverse square law, gravity, for example, we have the case $n = 1$ of the formulae above. Then

$$\overline{T} = -\frac{1}{2}\,\overline{V}$$

with

$$\overline{V} < 0$$

for a bound system.

It is instructive to check the virial theorem for a two-body system in a circular orbit. After the reduction to a one-body problem, the equation of motion is

$$\mu \ddot{\vec{r}} = -\hat{r}\, F(r)$$

For a circular orbit,

$$\ddot{\vec{r}} = -\hat{r}\,\frac{v^2}{r} = -\hat{r}\,\frac{|\dot{\vec{r}}|^2}{r}$$

so

$$\mu |\dot{\vec{r}}|^2 = r\, F(r)$$

which is just the theorem in its first form.

Another way to make the check is to use the effective potential energy for the 1-body problem discussed at the beginning of this lecture. This effective potential energy is, again,

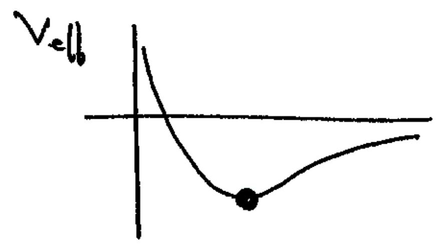
$$V_{eff} = \frac{L^2}{2\mu r^2} + V(r)$$

with

$$T = \frac{L^2}{2\mu r^2} \qquad V = V(r) = -|V(r)|$$

A circular orbit has $\dot{r} = 0$, and so it must lie at the minimum of this potential. If

$$V = -\frac{A}{n r^n}$$



the minimization problem is of the form

$$\frac{a}{r^2} - \frac{B}{r^n}$$

and the minimum is found where

$$\frac{2}{r}\left(\frac{a}{r^2}\right) - \frac{n}{r}\left(\frac{B}{r^n}\right) = 0$$

From this, it follows immediately that

$$T = \frac{n}{2}|V|$$

The virial theorem generalizes this balancing of kinetic and potential energy to any stable multi-particle system.

In astrophysics, the virial theorem can be used to estimate the masses of clusters of galaxies. The motion of the galaxies relative to one another along the line of sight is given by their relative Doppler shifts. This measures a contribution to the internal kinetic energy. To go further, we must assume, first, that the contributions to the kinetic energy from the other two axes are the same, and that the velocities are measured at a typical time at which kinetic and potential energy are balanced. With these assumptions, though, one can infer the strength of the gravitational potential in which the galaxies move and, from this, the total mass of the cluster. In 1933, Fritz Zwicky used this technique to argue that clusters of galaxies contain huge amounts of invisible mass – dark matter.

Finally, I will analyze the particular case of two particles bound by an inverse-square force, for example, a Coulomb or gravitational force. After reduction to a 1-body problem, the basic equation of motion is

$$\ddot{\vec{r}} = -\hat{r}\,\frac{g}{r^2}$$

where, specifically for the case of gravity,

$$g = \frac{G m_1 m_2}{\mu} = G(m_1 + m_2)$$

There are two famous ways to simplify this equation. First, consider the equation for the orbit in the plane perpendicular to the conserved $\vec{L}$. Let

$$c = |\vec{r} \times \dot{\vec{r}}| = \frac{|\vec{L}|}{\mu}$$

that is, the magnitude of L/μ for the reduced problem. If we use the coordinates (r, ϕ), then

$$c = r^2 |\dot{\phi}|$$

Since c is constant, we know how fast the particle will orbit at any radius, and so we only need to work out an equation for $r(t)$. To do this, compute $\dot{r}$

$$\dot{r} = \dot{\vec{r}} \cdot \hat{r} = \frac{\dot{\vec{r}} \cdot \vec{r}}{r}$$

and $\ddot{r}$,

$$\ddot{r} = \frac{\ddot{\vec{r}} \cdot \vec{r}}{r} + \frac{(\dot{\vec{r}})^2}{r} - \frac{(\vec{r} \cdot \dot{\vec{r}})^2}{r^3}$$

$$= \frac{\ddot{\vec{r}} \cdot \vec{r}}{r} + \frac{[(\dot{\vec{r}})_\perp]^2}{r}$$

where

$$(\dot{\vec{r}})_\perp = \dot{\vec{r}} - \hat{r}\,\hat{r} \cdot \dot{\vec{r}}$$

Since

$$|(\dot{\vec{r}})_\perp| = r\,\dot{\phi} = \frac{c}{r}$$

we have finally

$$\ddot{r} = -\frac{g}{r^2} + \frac{c}{r^3}$$

This equation for the variation of r can be converted to an orbit equation. Since

$$\dot{\phi} = \frac{c}{r^2}$$

we can relate

$$\frac{d}{dt} = \frac{c}{r^2} \frac{d}{d\phi}$$

Then

$$\frac{dr}{dt} = \frac{c}{r^2} \frac{dr}{d\phi} = -c \frac{d}{d\phi}\left(\frac{1}{r}\right)$$

$$\frac{d^2r}{dt^2} = -\frac{c^2}{r^2} \frac{d^2}{d\phi^2}\left(\frac{1}{r}\right)$$

Using this identity in the equation above, we find

$$-\frac{c^2}{r^2} \frac{d}{d\phi^2}\left(\frac{1}{r}\right) = -\frac{g}{r^2} + \frac{c}{r^3}$$

Setting

$$\varrho = \frac{1}{r}$$

converts this equation to

$$\frac{d^2}{d\phi^2} \varrho = \frac{g}{c^2} - \varrho$$

This is just the equation of a harmonic oscillator with $\omega^2 = 1$. We can immediately write down the general solution,

$$\varrho = \frac{g}{c^2} + A \cos(\phi - \phi_0)$$

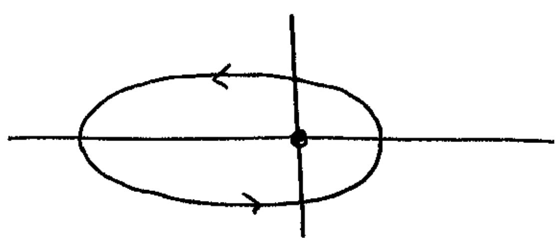
This is more conventionally written

$$r = \frac{c^2/g}{1 + e \cos f}$$

with

$$f = \phi - \phi_0$$

This is this equation for an ellipse with eccentricity e. For $\phi_0 = 0$, $e \neq 0$,



If $A = 0$ or $e = 0$, the ellipse collapses to a circle.

The semimajor axis of the ellipse is given by

$$2a = \frac{c^2/g}{1+e} + \frac{c^2/g}{1-e}$$

or

$$a = \frac{c^2/g}{1-e^2}$$

Then the equation for r can be rewritten as

$$r = \frac{a(1-e^2)}{1+e\cos f}$$

In planetary physics, f is called the *true anomaly.*

The frequency condition $\omega^2 = 1$ guarantees that that the orbit closes on itself. This condition is very specific to the case of inverse square forces.

There is another way to see that motion in an inverse square force field leads to closed orbits. It can be seen that this system has an extra constant of the motion, called the *Runge-Lenz vector.* (It was discovered by Hermann in the seventeenth century and was known to Laplace and Hamilton, among others, before it appeared in Runge's textbook.) This vector is defined by

$$\vec{\mathcal{R}} = \frac{\dot{\vec{r}} \times \vec{L}}{\mu} - g\hat{r}$$

We can check that $\vec{\mathcal{R}}$ is conserved,

$$\frac{d}{dt}\vec{\mathcal{R}} = \ddot{\vec{r}} \times (\vec{L}/\mu) - g\frac{(\dot{\vec{r}})_\perp}{r}$$

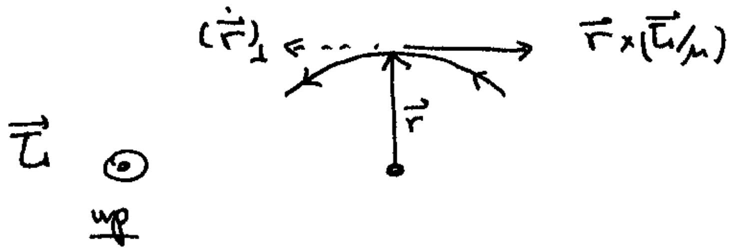
where

$$(\dot{\vec{r}})_\perp = \dot{\vec{r}} - \hat{r}\,\hat{r}\cdot\dot{\vec{r}}$$

Then

$$\frac{d}{dt}\vec{\mathcal{R}} = -\frac{g}{r^2}\hat{r} \times \frac{\vec{L}}{\mu} - g\frac{(\dot{\vec{r}})_\perp}{r}$$

The quantity $\hat{r} \times (\vec{L}/\mu)$ points in a direction perpendicular to $\vec{r}$ and to $\vec{L}$,



Its direction is opposite to that of $(\dot{\vec{r}})_\perp$. Its magnitude is

$$\left| \hat{r} \times \frac{\vec{L}}{\mu} \right| = r^2 \dot{\phi} = r \left| (\dot{\vec{r}})_\perp \right|$$

Thus, the two contributions to $d\vec{R}/dt$ cancel, and so

$$\frac{d}{dt} \vec{R} = 0$$

By construction, $\vec{R}$ is perpendicular to $\vec{L}$ and so lies in the plane of the motion. We can recognize that

$$\begin{aligned} \vec{r} \cdot \vec{R} &= \vec{r} \cdot (\dot{\vec{r}} \times \vec{L}/\mu) - g r \\ &= (\vec{r} \times \dot{\vec{r}}) \cdot \vec{L}/\mu - g r \\ &= c^2 - g r \end{aligned}$$

If we write

$$\vec{r} \cdot \vec{R} = r |R| \cos f$$

and parametrize

$$|\vec{R}| = ge$$

we find again the equation

$$r\, e \cos f = c^2/g - r$$

or

$$r = \frac{c^2/g}{1 + e \cos f}$$

as we found from the orbit equation above.

Real planets and moons do not move in perfect elliptical orbits, because they are perturbed by additional bodies in the solar system. We will study some of these perturbation problems later in the course.