Nov. 2

Canonical Transformations

In the previous lecture, I introduced Hamiltonian mechanics. I derived the equations of motion in Hamiltonian form, and I showed that these equations describe a Lagrangian system with n degrees of freedom as a special type of flow in a $2n$-dimensional phase space.

It was natural to ask what changes of variables on phase space preserve this Hamiltonian structure. We characterized these by the criterion that the Jacobian of such a change of variables is a real symplectic matrix. Formally, this is a requirement that, for the change of variables

$$x_i \rightarrow y_i(x)$$

the Jacobian

$$\mathcal{J}_{ij} = \frac{\partial y_i}{\partial x_j}$$

satisfies

$$\mathcal{J} E \mathcal{J}^T = E$$

A slightly more intuitive version of this constraint is that, for a continuously generated transformation, the generator satisfies

$$\mathcal{K} E = (\mathcal{K} E)^T$$

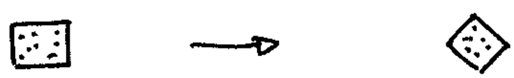
I apologize that neither statement is very visualizable.

In this lecture, I will try to remedy this in two ways. First, I will give some examples of simple symplectic transformations that will help you visualize how these transformations act on the phase space. Then I will introduce some formal tools for constructing canonical transformations and give examples of their application.

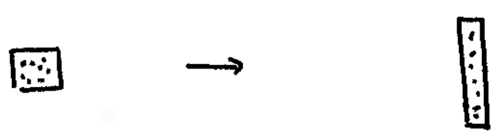
To begin, think about a phase space of 2 dimensions. The area element in this space is

$$dq\,dp$$

Symplectic transformations preserve the area, and we can visualize the class of transformations that satisfy this constraint. Intuitively these transformations correspond locally either to a rotation



or a stretching in one dimension and a compression in the other



or some combination of these operations.

This intuition can be generalized to symplectic transformations in higher dimensions. One expression of this is given by the *Poincare integral.* Let S be an element of 2-dimensional area in the $2n$-dimensional phase space, parametrized by coordinates α, β. The Poincare integral is the quantity

$$I = \int_S d\alpha\, d\beta \; \sum_i \left(\frac{\partial q_i}{\partial \alpha} \frac{\partial p_i}{\partial \beta} - \frac{\partial p_i}{\partial \alpha} \frac{\partial q_i}{\partial \beta} \right) = \int_S d\alpha\, d\beta \; \frac{\partial x_i}{\partial \alpha} E_{ij} \frac{\partial x_j}{\partial \beta}$$

If we change variables from x to y, the integral takes the value

$$I = \int d\alpha\, d\beta \quad \frac{\partial y_k}{\partial \alpha} \underbrace{\frac{\partial x_i}{\partial y_k}}_{(\mathcal{J}^{-1})^T} E_{ij} \underbrace{\frac{\partial x_j}{\partial y_l}}_{\mathcal{J}^{-1}} \frac{\partial y_l}{\partial \beta}$$

As long as

$$\mathcal{J} E \mathcal{J}^T = E \qquad \rightarrow \qquad \mathcal{J}^{-1} = -E \mathcal{J}^T E$$

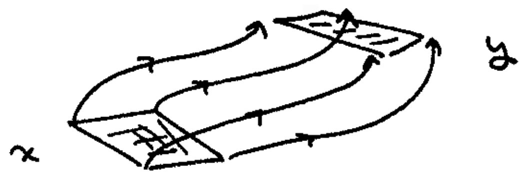
the matrix inside the integrand becomes

$$\begin{aligned} (\mathcal{J}^{-1})^T E \mathcal{J}^{-1} &= E \mathcal{J} E E E \mathcal{J}^T E = - E \mathcal{J} E \mathcal{J}^T E \\ &= E \end{aligned}$$

and so

$$I = \int d\alpha\, d\beta \quad \frac{\partial y_k}{\partial \alpha} E_{kl} \frac{\partial y_l}{\partial \beta}$$

That is, the Poincare integral over S is equal to the Poincare integral over the image of S under a canonical transformation.



Even in the full $2n$-dimensional phase space, the area of any 2-dimensional surface is preserved under a canonical transformation.

Another way to look at canonical transformations is to think about the eigenvalues of the Jacobian. These are most easily analyzed by looking at the eigenvalues of the infinitesimal generators. Write, again,

$$\mathcal{J} = 1 + \mathcal{K} \, \Delta\alpha + \cdots$$

where $\Delta\alpha$ is the infinitesimal parameter. Let k be an eigenvalue of $\mathcal{K}$. Then k is a solution of the characteristic equation

$$\det(\underline{K} - k\underline{1}) = 0$$

Since $\mathcal{K}$ is a real matrix, we can complex-conjugate this equation and find

$$\det(\underline{K} - k^{*}\underline{1}) = 0$$

If k is a eigenvalue, so is k^*. Now

$$\det E = 1 \neq 0$$

so we do not change the equation if we multiply by $\det E$. Then

$$\det (K E - k E) = 0$$

Since $\mathcal{K}$ satisfies

$$K E = (K E)^T$$

it follows that

$$\det ((K E)^T - k E) = 0$$

or

$$\det (E^T K^T + E^T k) = 0$$

which implies

$$\det (\underline{K}^T + k \underline{1}) = 0$$

Thus, $(-k)$ is an eigenvalue of $\mathcal{K}$. We have now shown that if k is an eigenvalue of $\mathcal{K}$, so are

$$k , \quad k^* , \quad -k , \quad -k^*$$

Thus, eigenvalues of $\mathcal{K}$ come in either pairs or quartets.

There are three distinct cases:

1. k is pure imaginary. In this case, $\mathcal{K}$ has eigenvalues

$$i\lambda \qquad -i\lambda$$

A real matrix $\mathcal{K}$ with these eigenvalues has the form

$$\underline{K} = \begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix}$$

This infinitesimal transformation integrates to a finite transformation

$$e^{\underline{K}t} = \begin{pmatrix} \cos\lambda t & -\sin\lambda t \\ \sin\lambda t & \cos\lambda t \end{pmatrix}$$

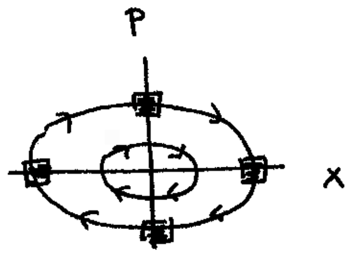
which is a local rotation. This behavior is seen in the simple harmonic oscillator, for which the Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

The Hamiltonian equations are

$$\dot{x} = \frac{p}{m} \qquad \dot{p} = -m\omega^2 x$$

This generates the flows in the 2-dimensional phase space



Each small element of area is rotated without distortion.

2. k is pure real. In this case, $\mathcal{K}$ has eigenvalues

$$\lambda \qquad -\lambda \qquad\qquad \underline{K} = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}$$

This infinitesimal transformation integrates to a finite transformation

$$e^{\underline{K}t} = \begin{pmatrix} \cosh\lambda t & \sinh\lambda t \\ \sinh\lambda t & \cosh\lambda t \end{pmatrix}$$

which is stretching along one axis and compression along an orthogonal axis. This behavior is seen, for example, in the oscillator with an unstable potential, with the Hamiltonian

$$\mathcal{H} = \frac{p^2}{2m} - \frac{1}{2} m\omega^2 x^2$$

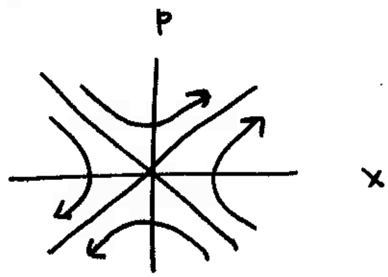
The Hamiltonian equations of motion are

$$\dot{x} = \frac{p}{m} \qquad \dot{p} = -m\omega^2 x$$

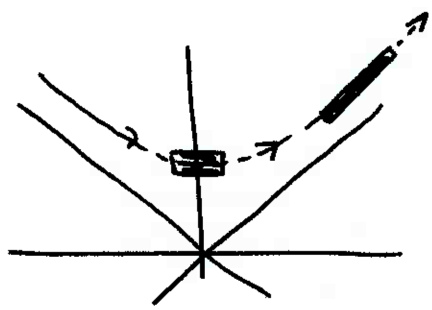
The solutions to these equations are

$$x = A e^{\omega t} + B e^{-\omega t} \qquad p = A\omega e^{\omega t} - B\omega e^{-\omega t}$$

This generates flows in the 2-dimensional phase space



In this flow, an initial element of area is stretched along the motion and squeezed perpendicular to the motion,



3. k is complex. In this case, $\mathcal{K}$ has a subspace with the four eigenvalues

$$\alpha + i\beta \ , \quad \alpha - i\beta \ , \quad -\alpha + i\beta \ , \quad -\alpha - i\beta$$

$$\alpha > 0$$

To represent such a $\mathcal{K}$, we need a 4×4 matrix of real elements, of the form

$$\underline{K} = \begin{pmatrix} B^T & 0 \\ 0 & -B \end{pmatrix}$$

where B is a 2×2 matrix with eigenvalues $\alpha \pm i\beta$. Regions of phase space are stretched and rotated in the upper two coordinates and compressed and rotated in the opposite direction in the lower two coordinates.

It can be shown that any matrix $\mathcal{K}$ that is a generator of symplectic transformations can be reduced by a real change of coordinates to block diagonal form, which each block is a 2×2 or a 4×4 with one of the forms above. Thus, the three simple structures just discussed contain the generic situation, and we can use them to visualize the the changes in phase space volumes induced by a general canonical transformation.

Now that we have made some progress in understanding canonical transformations geometrically, I will introduce some tools for understanding them better analytically.

First, I will introduce a very useful formal object. Let $A(q,p)$ and $B(q,p)$ be two general functions on phase space. Define the *Poisson bracket* of A and B by

$$\{A,B\} = \sum_i \left(\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right)$$

$$= \frac{\partial A}{\partial x_i} E_{ij} \frac{\partial B}{\partial x_j}$$

This object is invariant to canonical transformations. Under a canonical transformation $x \to y$,

$$\frac{\partial B}{\partial x_j} = \frac{\partial y_l}{\partial x_j} \frac{\partial B}{\partial y_l} = \left(\mathcal{J}^T \frac{\partial B}{\partial y} \right)_j$$

Then

$$\frac{\partial A}{\partial x_i} E_{ij} \frac{\partial B}{\partial x_j} = \left(\frac{\partial A}{\partial y} \right)^T \mathcal{J} E \mathcal{J}^T \frac{\partial B}{\partial y}$$

and so

$$\left(\frac{\partial A}{\partial x} \right)^T E \frac{\partial A}{\partial x} = \left(\frac{\partial A}{\partial y} \right)^T E \frac{\partial B}{\partial y}$$

Thus, we can compute the Poisson bracket in any set of coordinates on phase space that are canonically related to the original coordinates (q,p).

Note that the q_i an p_j have a very simple Poisson bracket with respect to one another. The Poisson bracket of q_i is just

$$\{q_i, q_j\} = \sum_k \left(\underbrace{\frac{\partial q_i}{\partial q_k}}_{\delta_{ik}} \underbrace{\frac{\partial q_j}{\partial p_k}}_{0} - \underbrace{\frac{\partial q_i}{\partial p_k}}_{0} \underbrace{\frac{\partial q_j}{\partial q_k}}_{\delta_{jk}} \right) = 0$$

The bracket of q_k with p_ℓ is

$$\{q_k, p_\ell\} = \sum_i \left(\frac{\partial q_k}{\partial q_i} \frac{\partial p_\ell}{\partial p_i} - \frac{\partial q_k}{\partial p_i} \frac{\partial p_\ell}{\partial q_i} \right) = \delta_{ki}\delta_{\ell i} - 0 = \delta_{k\ell}$$

In all, we find

$$\{q_i, q_j\} = 0 \qquad \{q_i, p_j\} = \delta_{ij} \qquad \{p_i, p_j\} = 0$$

These relations are called the *canonical Poisson brackets.*

Now we can use the Poisson bracket to express the Hamiltonian equations more algebraically. First, note that

$$\begin{aligned} \{q_j, H\} &= \sum_i \left(\frac{\partial q_j}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial q_j}{\partial p_i} \frac{\partial H}{\partial q_i} \right) = \delta_{ji} \frac{\partial H}{\partial p_i} - 0 \\ &= \frac{\partial H}{\partial p_j} \end{aligned}$$

Similarly,

$$\{p_j, H\} = -\frac{\partial H}{\partial q_j}$$

Thus,

$$\dot{q}_j = \{q_j, H\} \qquad \dot{p}_j = \{p_j, H\}$$

So the Hamiltonian equations of motion take a simple form in terms of Poisson brackets.

By the chain rule, this equation of time evolution applies to any function of q and p,

$$\dot{A}(q,p) = \{A, H\}$$

If $A(q,p)$ is a distribution on phase space, this equation gives its time evolution. In particular, if $C(q,p)$ is a constant of the motion,

$$\{C(q,p), H\} = 0$$

An important special case is that in which a coordinate ϕ does not appear in H. Then the conjugate momentum satisfies

$$\{p_\phi, H\} = 0$$

and so is a constant of the motion. This is the expression of Noether's theorem in Hamiltonian language.

The Poisson bracket is an antisymmetric structure similar to the commutator of operators. In fact, the Poisson bracket shares many properties with the commutator. In the lecture on Lie groups, we saw that the double commutator obeys a general property called the Jacobi identity. The Poisson bracket obeys the same property,

$$\{A,\{B,C\}\} + \{B,\{C,A\}\} + \{C,\{A,B\}\} = 0$$

The proof is the same as for commutators: Write out all of the terms on a large piece of paper – there are 24 terms in all – and notice that they cancel pairwise. For example, the first term is

$$\frac{\partial A}{\partial q_i}\frac{\partial}{\partial p_i}\left(\frac{\partial B}{\partial q_j}\right)\frac{\partial C}{\partial p_j}$$

while the last double bracket contains

$$-\frac{\partial C}{\partial p_j}\frac{\partial A}{\partial q_i}\frac{\partial}{\partial q_j}\left(\frac{\partial B}{\partial p_i}\right)$$

Among many consequences of the Jacobi identify is the following: If C and D are constants of the motion, so is $\{C, D\}$. Here is the proof:

$$\{\{C,D\},H\} = -\{H,\{C,D\}\}$$

$$= \{C,\underbrace{\{D,H\}}_{0}\} + \{D,\underbrace{\{H,C\}}_{0}\} = 0$$

We describe this result by saying that the conserved quantities form a *closed algebra* under the Poisson bracket.

Now that we have studied the structure of canonical transformations, the next question is, how do we actually construct canonical transformations. In particular, if we wish to make a change of variables

$$q_i \to Q_i(q)$$

what is the transformation

$$p_i \rightarrow P_i(p,q)$$

that makes the transformation a canonical change of variables?

There is a formal way to make this change of variables in such a way that the transformation is guaranteed to be canonical. This is the method of *generating functions.* The method is related to the constancy of the Poincare invariant under canonical changes of variables. For the change of variables

$$(q_i p_i) \longrightarrow (Q_i, P_i)$$

we showed that

$$\frac{\partial q_i}{\partial \alpha}\frac{\partial p_i}{\partial \beta} - \frac{\partial p_i}{\partial \alpha}\frac{\partial q_i}{\partial \beta} = \frac{\partial Q_i}{\partial \alpha}\frac{\partial P_i}{\partial \beta} - \frac{\partial P_i}{\partial \alpha}\frac{\partial Q_i}{\partial \beta}$$

Now we can rearrange this equation into the form

$$\frac{\partial}{\partial \beta}\left(p_i \frac{\partial q_i}{\partial \alpha} - P_i \frac{\partial Q_i}{\partial \alpha}\right) = \frac{\partial}{\partial \alpha}\left(p_i \frac{\partial q_i}{\partial \beta} - P_i \frac{\partial Q_i}{\partial \beta}\right)$$

or

$$\frac{\partial}{\partial \alpha} V_\beta - \frac{\partial}{\partial \beta} V_\alpha = 0$$

where

$$V_\alpha = p_i \frac{\partial q_i}{\partial \alpha} - P_i \frac{\partial Q_i}{\partial \alpha}$$

These formulae are reminiscent of the equations that relate the vector potential and the field strength in electrodynamics. We have shown, effectively, that the vector potential V_α is such that the corresponding field strength is zero. In another language, we have shown that the generalized curl of the vector field V_α is zero. One way or the other, this implies that the vector field can be written, locally, as a gradient, or, in electrodynamics language, a pure gauge. That is,

$$V_\alpha = \frac{\partial}{\partial \alpha} F_1$$

The scalar function F_1 is a function of phase space coordinates. We can write it as a function of q and p, but it is very convenient to eliminate p and write it as a function of q and Q.

$$F_1(q, Q)$$

Then

$$V_\alpha = \frac{\partial}{\partial \alpha} F_1 = \frac{\partial F_1}{\partial q_i} \frac{\partial q_i}{\partial \alpha} + \frac{\partial F_1}{\partial Q_i} \frac{\partial Q_i}{\partial \alpha}$$

But, from the equation above,

$$V_\alpha = p_i \frac{\partial q_i}{\partial \alpha} - P_i \frac{\partial Q_i}{\partial \alpha}$$

Since

$$\left(\frac{\partial q_i}{\partial \alpha}, \frac{\partial Q_i}{\partial \alpha} \right)$$

should be independent vectors in phase space, we can read off

$$p_i = \frac{\partial F_1}{\partial q_i} \qquad -P_i = \frac{\partial F_1}{\partial Q_i}$$

Thus, associated with each canonical transformation, there is a *generating function* $F_1(q, Q)$ such that

$$p_i = \left.\frac{\partial F_1}{\partial q_i}\right|_Q \qquad -P_i = \left.\frac{\partial F_1}{\partial Q_i}\right|_q$$

Conversely, we can take the transformation to be defined by a function $F_1(q, Q)$, with the variables p and P defined by these equations. Then, reversing the argument, we will find that

$$\frac{\partial q_i}{\partial \alpha}\frac{\partial p_i}{\partial \beta} - \frac{\partial p_i}{\partial \alpha}\frac{\partial q_i}{\partial \beta} = \frac{\partial Q_i}{\partial \alpha}\frac{\partial P_i}{\partial \beta} - \frac{\partial P_i}{\partial \alpha}\frac{\partial Q_i}{\partial \beta}$$

If we now write

$$x = (q, p) \qquad y = (Q, P)$$

this implies

$$\frac{\partial x_i}{\partial \alpha} E_{ij} \frac{\partial x_j}{\partial \beta} = \frac{\partial y_i}{\partial \alpha} E_{ij} \frac{\partial y_j}{\partial \beta}$$

for any α and β. Thus, the transformation we have defined is canonical.

Actually, a canonical transformation has not one but four generating functions. The second one is found by recasting the relation of the invariance of the Poincare integral as

$$\frac{\partial}{\partial \beta}\left(p_i \frac{\partial q_i}{\partial \alpha} + Q_i \frac{\partial P_i}{\partial \alpha} \right) = \frac{\partial}{\partial \alpha}\left(p_i \frac{\partial q_i}{\partial \beta} + Q_i \frac{\partial P_i}{\partial \beta} \right)$$

This implies

$$V_\alpha = \frac{\partial}{\partial \alpha} F_2 = p_i \frac{\partial q_i}{\partial \alpha} + Q_i \frac{\partial P_i}{\partial \alpha}$$

Then, if we write F_2 as a function of q, P, we can generate the canonical transformation from

$$\frac{\partial F_2}{\partial q_i} = p_i \qquad \frac{\partial F_2}{\partial P_i} = Q_i$$

The relation between the derivatives of F_1 and F_2 is just that induced by the Legendre transformation

$$F_2 = F_1 + P_i Q_i$$

Similarly, we can transform to a generating function written in terms of p and Q,

$$F_3 = F_1 - p_i q_i$$

which gives

$$\frac{\partial F_3}{\partial p_i} = -q_i \qquad \frac{\partial F_3}{\partial Q_i} = -P_i$$

or a generating function written in terms of p and P,

$$F_4 = F_2 - p_i q_i$$

which gives

$$\frac{\partial F_4}{\partial p_i} = -q_i \qquad \frac{\partial F_4}{\partial P_i} = Q_i$$

We have seen at least one of these functions before. Recall that the action integral for a motion from the initial conditions q_i to the final conditions Q_i along the path that extremizes

$$S(q, Q, T)$$

satisfies the relations

$$\left.\frac{\partial S}{\partial q_i}\right|_Q = -p_i \qquad \left.\frac{\partial S}{\partial Q_i}\right|_q = P_i$$

Then, for time evolution through T, which is a canonical transformation,

$$F_1(q, Q; T) = -S(q, Q; T)$$

Here is an example of the use of a generating function to construct a canonical transformation that solves a problem of mechanics. The problem will be the very simple one of a harmonic oscillator

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2$$

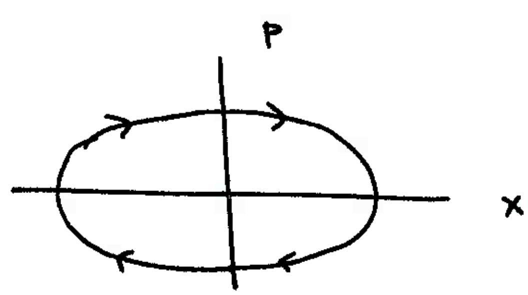
The equations of motion are

$$\dot{x} = \frac{p}{m} \qquad \dot{p} = -m\omega^2 x$$

and we already know the form of the general solution

$$x = A \cos(\omega t + \phi_0) \qquad p = -A m \omega \sin(\omega t + \phi_0)$$

At the beginning of the lecture, we worked out the phase space flows, which we saw follow ellipses



I would like to solve this problem by trading the coordinate x for a new coordinate Q that parametrizes the phase of the oscillation. An appropriate generating function is

$$F_1 = -\frac{m\omega x^2}{2} \tan Q$$

From this, we see that

$$p = \frac{\partial F_1}{\partial x} = -m\omega\, x \tan Q$$

so that

$$\tan Q = \frac{p}{-m\omega x} = \frac{\sin(\omega t+\phi_0)}{\cos(\omega t+\phi_0)}$$

Thus, the new variable Q is indeed the one that we were seeking. Notice that Q is periodic with period 2π.

Let P be the momentum conjugate to Q. This phase space coordinate P should be completely fixed by the requirement that the transformation be canonical. We can compute P from $F_1(x, Q)$.

$$P = -\frac{\partial F_1}{\partial Q} = \frac{m\omega x^2}{2} \frac{1}{\cos^2 Q}$$

Solving for x, we find

$$x = \left(\frac{2P}{m\omega}\right)^{1/2} \cos Q$$

From the above, we find for p

$$p = -m\omega \left(\frac{2\mathcal{P}}{m\omega}\right)^{1/2} \cos Q \cdot \tan Q = -(2m\omega\mathcal{P})^{1/2} \sin Q$$

Plugging these results into the Hamiltonian, we find

$$\mathcal{H} = \frac{1}{2m} \; 2m\omega \mathcal{P} \sin^2 Q + \frac{1}{2} m\omega^2 \frac{2\mathcal{P}}{m\omega} \cos^2 Q$$

That is

$$\mathcal{H} = \omega \mathcal{P}$$

Since Q is a simple angle, it is conventional to call it ϕ and to call the conjugate momentum J.

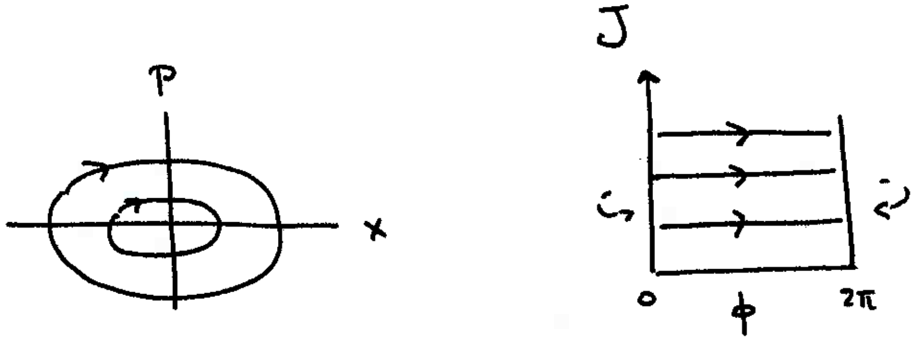
The new Hamiltonian is then

$$\mathcal{H} = \omega J$$

In the new coordinates, the Hamiltonian is even simpler than before. The equations of motion are

$$\dot{\phi} = \omega \qquad\qquad \dot{J} = 0$$

The momentum J is a conservation law, and the coordinate ϕ experiences only a uniform translation with time. The flow pattern is mapped to a cylinder, periodically connected under $\phi \to \phi + 2\pi$:



Notice that J is laid out on the cylinder in such a way that the mapping preserves area.

We can apply the idea of generating functions to solve the problem put forward earlier in the lecture: Given a desired change of variables

$$q_i \longrightarrow Q_i(q) = \mathcal{Q}_i(q)$$

where $\mathcal{Q}_i(q)$ are given functions of the q_i, can we find a corresponding change of variables for the p_i

$$p_i \longrightarrow P_i(q, p)$$

so that the transformation is canonical? To solve this problem, use $F_2(q, P)$. Write

$$F_2(q, P) = P_i \mathcal{Q}_i(q)$$

Then

$$Q_i = \frac{\partial F_2}{\partial P_i} = \mathcal{Q}_i(q)$$

as required. From the same function, we find

$$p_i = \frac{\partial F_2}{\partial q_i} = \frac{\partial \mathcal{Q}_j(q)}{\partial q_i} P_j$$

If we let

$$\mathcal{M}_{ij} = \frac{\partial}{\partial q_i} \mathcal{Q}_j(q)$$

then

$$P_j = (\mathcal{M}^{-1}(q))_{jk}\, p_k$$

The transformation $(q, p) \to (Q, P)$ is a canonical transformation. Notice that it manifestly preserves the volume in phase space.

We can also use the language of generating functions to discuss infinitesimal canonical transformations. The formalism we have build applies to finite canonical transformations, but it also applies to small transformations, provided that the two sets of coordinates being related do not become degenerate. This is a problem if we use $F_1(q, Q)$, since $q_i = Q_i$ for the identity transformation, so that the coordinates (q, Q) are not good coordinates for phase space. To avoid this problem, I will work with $F_2(q, P)$. The generating function

$$F_2 = q_i P_i$$

gives the identity transformation. Explicitly,

$$p_i = \frac{\partial F_2}{\partial q_i} = P_i \qquad\qquad Q_i = \frac{\partial F_2}{\partial P_i} = q_i$$

so

$$(q_i, p_i) = (Q_i, P_i)$$

To describe a continuously generated transformation, we write a function $F_2(q, P)$ that tends to this function as $\alpha \to 0$. Then

$$F_2(q, P) = q_i P_i + \Delta\alpha \; G(q, P) + \cdots$$

From this form of the generating function, the infinitesimal transformation is

$$Q_i = q_i + \Delta\alpha \frac{\partial G}{\partial P_i} + \cdots$$

and

$$p_i = P_i + \Delta\alpha \frac{\partial G}{\partial q_i} + \cdots$$

or

$$P_i = p_i - \Delta\alpha \frac{\partial G}{\partial q_i} + \cdots$$

We can think of $G(q, p)$ as a function on phase space that generates the transformation according to

$$\frac{dq_i}{d\alpha} = \frac{\partial G}{\partial p_i} \qquad \frac{dp_i}{d\alpha} = -\frac{\partial G}{\partial q_i}$$

These equations are conveniently written using the Poisson bracket as

$$\frac{d}{d\alpha} q_i = \{q_i, G\} \qquad \frac{d}{d\alpha} p_i = \{p_i, G\}$$

or, if $A(q,p)$ is any distribution on the phase space

$$\frac{d}{d\alpha} A = \{A, G\}$$

The function $G(q,p)$ plays a role here similar to the Hamiltonian. Just as the Hamiltonian generates the canonical transformation of time translation, G generates a more arbitrary continuous transformation.

A particular special case is

$$G = p_k$$

The canonical Poisson bracket relations above imply

$$\frac{dq_i}{d\alpha} = \{q_i, p_k\} = \delta_{ik} \qquad \frac{dp_i}{d\alpha} = \{p_i, p_k\} = 0$$

so this form of G generates a simple translation of the coordinate q_k. On a distribution,

$$\frac{d}{d\alpha} A = \{A, p_k\} = \frac{\partial A}{\partial q_k}$$

In a mechanical system of particles, the momentum $\vec{p} = m\dot{\vec{x}}$ generates translations of $\vec{x}$. But also, the components of angular momentum are conjugate momenta. The angular momentum component L^z, for example, generates translations of the polar angle ϕ about the $\hat{z}$ axis.

Many of our results for the Poisson bracket are reminiscent of result for the commutator of operators. In quantum mechanics, these results are used to build up a set of dynamical equations analogous to those in Hamiltonian mechanics. We define Hermitian operators q_i and p_j and assign these the *canonical commutation relations*

$$[q_i, q_j] = 0 \qquad [q_i, p_j] = i\delta_{ij}\hbar \qquad [p_i, p_j] = 0$$

analogous to the canonical Poisson brackets above. We then assign these operators the equations of motion

$$\dot{q}_i = \frac{-i}{\hbar}[q_i, H] \qquad \dot{p}_i = \frac{-i}{\hbar}[p_i, H]$$

analogous to the Hamiltonian equations. The q_i and p_j in Hamiltonian mechanics and in quantum mechanics are different types of mathematical objects living in completely different mathematical spaces. But, for some purposes, we do not need to know the identity or internal structure of these objects. Often, we can work out the explicit equations of motion directly from the canonical commutation relations or Poisson brackets by using the chain rule

$$\{ab, H\} = a\{b, H\} + \{a, H\}b$$

which is correct both for Poisson brackets and for commutators. We obtain the same equations of motion if we use the same form of the Hamiltonian and replace the Poisson bracket by the commutator according to

$$\{\ ,\ \} \longrightarrow \frac{-i}{\hbar}[\ ,\ ]$$

The idea of operators generating symmetry transformations also goes over from Hamiltonian mechanics to quantum mechanics. A symmetry transformation in quantum mechanics is generated by a Hermitian operator G according to

$$\frac{d}{d\alpha} A = \frac{-i}{\hbar} [A, G]$$

Then the transformation takes the form of a unitary operator applied to wavefunctions

$$|\psi\rangle \rightarrow e^{-\frac{i}{\hbar} G\alpha} |\psi\rangle$$

or a unitary action on operators

$$A \rightarrow e^{\frac{i}{\hbar} G\alpha} A \, e^{-\frac{i}{\hbar} G\alpha}$$

Again, $G = H$ generates time translations, $G = \hat{n} \cdot \vec{p}$ generates spatial translations, and $G = \hat{n} \cdot \vec{L}$ generates spatial rotations.

Eventually, however, we must confront the fact that, while phase space coordinates can be known exactly, Hermitian operators do not commute with each other and give definite predictions only when acting on their eigenvectors. Exploration of these differences will lead us to a discussion of the mysteries of quantum mechanics, which is the subject of a different course.