

## Physics 152/252 – Problem Set # 7

(due Thursday, May 19)

1. A field theory with spontaneous symmetry breaking can be constructed as follows: First, write the Klein-Gordon Lagrangian for  $n$  fields  $\phi_i$ ,

$$\mathcal{L} = \sum_i \left[ \frac{1}{2} (\partial_\mu \phi_i)^2 - \frac{1}{2} m_i^2 \phi_i^2 \right] \quad (1)$$

then replace the mass term by  $V(\phi)$ , a general nonlinear function of the  $\phi_i$ , to form

$$\mathcal{L} = \sum_i \left[ \frac{1}{2} (\partial_\mu \phi_i)^2 \right] - V(\phi) \quad (2)$$

The function  $V(\phi)$  is the potential energy associated with the scalar field value. A system with spontaneous symmetry breaking,  $V(\phi)$  has its minimum at a value  $\phi = \Phi$  that does not respect the symmetry of  $V(\phi)$ . This problem will explore some properties of this theory.

- (a) Show that the equations of motion of this theory are

$$\partial_\mu \partial^\mu \phi_i + \frac{\partial}{\partial \phi_i} V(\phi) = 0 \quad (3)$$

- (b) The minimum of the potential energy is a constant vector  $\Phi_i$  satisfying

$$\frac{\partial}{\partial \phi_i} V(\phi)|_{\phi=\Phi} = 0 \quad (4)$$

Writing

$$\phi_i(x) = \Phi_i + \eta_i(x) \quad (5)$$

expand the equations of motion up to terms of first order in  $\eta_i(x)$ . Show that the  $n$  eigenvalues of the matrix

$$M_{ij}^2 = \frac{\partial^2}{\partial \phi_i \partial \phi_j} V(\phi)|_{\phi=\Phi} \quad (6)$$

give  $n$  values of (mass)<sup>2</sup> corresponding to  $n$  scalar particles in the theory.

- (c) Consider the potential

$$V(\phi) = +\frac{1}{2} \mu^2 (\phi^2) + \frac{1}{4} \lambda (\phi^2)^2 \quad (7)$$

where  $\phi^2 = \sum_i (\phi_i)^2$  and  $\mu$  and  $\lambda$  are constants. Sketch this potential. Find the minimum of the potential. Find the masses of the  $n$  particles.

(d) Consider the potential

$$V(\phi) = -\frac{1}{2}\mu^2(\phi^2) + \frac{1}{4}\lambda(\phi^2)^2 \quad (8)$$

where everything is as before, except that I have changed the sign in front of  $\mu^2$ . Sketch this potential. Find the minimum of  $V$  among constant fields of the form

$$\phi = (0, 0, 0, \dots, 0, v) \quad (9)$$

Show that the minimum occurs for  $v \neq 0$ , and find the value of  $v$  at the minimum. This is spontaneous symmetry breaking. Show that this potential  $V$  has an  $(n-1)$ -dimensional sphere of degenerate minima.

(e) Find the masses of particles in this theory. Show that  $(n-1)$  of these masses are zero. This illustrates Goldstone's theorem.

2. The quark masses given by the Particle Data Group are "running masses in the  $\overline{MS}$  scheme". Without going into too much detail about the definition, I note that (1) like all other quantities in QCD, the quark masses evolve as functions of the momentum scale  $Q$ , and (2) therefore, quark masses must be quoted at a particular value of  $Q$ . The PDG values are:

flavor	$m_f(Q_0)$	$Q_0$	flavor	$m_f(Q_0)$	$Q_0$
$u$	0.0023	2	$c$	1.28	1.28
$d$	0.0048	2	$b$	4.18	4.18
$s$	0.095	2	$t$	164.	164.

(10)

with all mass values in GeV. In this problem, we will compare the quark masses in a more invariant way. Our analysis will be as simple as possible, to leading order in QCD only.

(a) In class, we found the following expression for  $\alpha_s$ :

$$\alpha_s(Q) = \frac{\alpha_s(Q_0)}{1 + (b_0\alpha_s(Q_0)/2\pi) \log(Q/Q_0)} \quad (11)$$

where  $b_0 = 11 - \frac{2}{3}n_f$  and  $n_f$  is the number of quark flavors with  $m_f < Q$ . Using  $\alpha_s(91.) = 0.118$ , evaluate  $\alpha_s(Q)$  at the  $Q$  values:  $m_b = 4.18$  GeV, 2 GeV, and  $m_c = 1.28$  GeV. Note that you will need to use different values of  $b_0$  for  $Q > m_b$  and  $Q < m_b$  to convert the  $\alpha_s$  values.

(b) QCD gives the following equation for the  $Q$ -dependence of a quark mass parameter,

$$\frac{d}{d \log Q} m_f(Q) = -8 \frac{\alpha_s(Q)}{4\pi} m_f(Q) \quad (12)$$

Using the formula for  $\alpha_s$  in (a), find the solution of this equation that gives  $m_f(Q)$  in terms of a reference value  $m_f(Q_0)$ .

- (c) Compute the value of the charm quark mass  $m_c$  at  $Q = 2$  GeV.
- (d) Compute the values of the four lightest quark masses at  $Q = m_b$ .
- (e) Compute the values of all quark masses at  $Q = m_t$  given above. You will need to find  $\alpha_s(m_t)$  from  $\alpha_s(91.)$  using (11).
- (f) Compute the true ratios of quark masses compared at this common value of  $Q$ .