Physics 152/252 – Problem Set # 1

(due Thursday, April 7)

1. Consider the decay of a particle of mass M, at rest, into two particles with masses m_1 and m_2, both nonzero. With an appropriate choice of axes, the momentum vectors of the final particles can be written

$$p_1 = (E_1, 0, 0, k) \qquad p_2 = (E_2, 0, 0, -k) \tag{1}$$

with $E_1^2 = k^2 + m_1^2$, $E_2^2 = k^2 + m_2^2$.

   (a) Show that

$$k = \Big[(M^4 - 2M^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2\Big]^{1/2} / 2M \tag{2}$$

   (b) Take the limit $m_2 \to 0$ and show that this reproduces the result for the decay into one massive and one massless particle, discussed in class.

   (c) Find formulae for E_1 and E_2 in terms of M, m_1, m_2.

2. Consider an event in which an unstable particle H decays into two photons. Work in the rest frame of the unstable particle. The photons are emitted back-to-back. Take the $\hat{3}$ axis to be aligned with the direction of the photons. Let photon 1 be the one travelling in the $+\hat{3}$ direction and photon 2 be the one travelling the $-\hat{3}$ direction.

   (a) Argue that the spin of H must be integer, not half-integer.

   (b) Possible polarization vectors for the photon 1 are

$$\vec{\epsilon}_{1R} = \frac{1}{\sqrt{2}}(\hat{1} + i\hat{2}) \qquad \vec{\epsilon}_{1L} = \frac{1}{\sqrt{2}}(\hat{1} - i\hat{2}) \tag{3}$$

   Rotate these vectors by ϕ about the $\hat{3}$ axis. A state of angular momentum $J^3 = +1$ gets a phase $e^{-i\phi}$. Show that the two choices correspond to photon states of angular momentum $J^3 = +1$ and -1, respectively, about the $\hat{3}$ axis.

   (c) Write the corresponding polarization vectors for photon 2, by rotating the vectors in (1) by 180° about $\hat{2}$. These have $J^3 = +1, -1$ about the direction of motion of the photon (which is now $-\hat{3}$).

   (d) The wave function of the 2-photon state is then a sum of terms of the form

$$\vec{\epsilon}_{1X} \vec{\epsilon}_{2Y} \tag{4}$$

   where $X, Y = R, L$. There are four possible values for (X, Y). For each, compute the total J^3 for the state (2). Show that, in the states with $X = R, Y = L$ or $X = L, Y = R$, the spin of the original particle H must be ≥ 2.

(e) Consider the state with $X = Y = R$. Show that this state is transformed into itself by a rotation by 180° about $\hat{2}$. The same is true for the state $X = Y = L$.

(f) If the original particle H has spin J and decays to the state $X = Y = R$, it must have been in the state $|J0\rangle$, with $J^3 = 0$. How does this state transform when rotated by 180° about $\hat{2}$? (The transformation must be the same as that of the spherical harmonic $Y_{J0}(\theta, \phi)$.)

(g) Conclude that an unstable particle of spin 1 may not decay to two photons. This result is called the Landau-Yang theorem. (Note that invariance under Parity has not been used in this argument.)

3. The Dirac matrices γ^μ, $\mu = 0, 1, 2, 3$, are 4×4 matrices that satisfy the algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \ ,$$

where $g^{\mu\nu}$ is the metric tensor of special relativity with (1, -1, -1, -1) on the diagonal.

(a) Show that the following set of matrices satisfies the Dirac algebra:

$$\gamma^0_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^i_A = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

Here each entry is a 2×2 matrix and σ^i, $i = 1, 2, 3$, are the Pauli sigma matrices.

(b) Show that the following set of matrices satisfies the Dirac algebra:

$$\gamma^0_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^i_B = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

(c) Show that these representations are equivalent. That is, write a 4×4 unitary matrix U such that

$$\gamma^\mu_B = U \gamma^\mu_A U^\dagger$$

It can be shown that all 4×4 representations of the Dirac algebra are equivalent by unitary transformations.