Physics 152/252

Problem Set #8 — Solutions

1.) a.) The decays $\tau \to \nu_\tau e \bar{\nu}_e$ and $\tau \to \nu_\tau \mu \bar{\nu}_\mu$ proceed through the V-A interaction in exactly the same way as $\mu \to \nu_\mu e \bar{\nu}_e$. So, the same formula applies!

$$\Gamma(\tau \to \nu_\tau e \bar{\nu}_e) = \Gamma(\tau \to \nu_\tau \mu \bar{\nu}_\mu) = \frac{G_F^2 m_\tau^5}{192\pi^3}$$

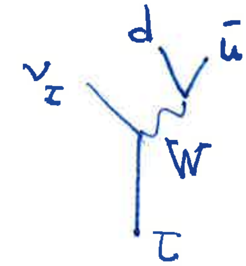
$$= 4.05 \times 10^{-13} \text{ GeV}$$

$$= 4.05 \times 10^{-10} \text{ MeV}$$

b.) For $\tau \to \nu_\tau d \bar{u}$, the calculation is again the same, except that we must add:

(1) the color factor 3

(2) the QCD correction $\left(1 + \frac{\alpha_s(m_\tau)}{\pi}\right)$



then

$\alpha_s(m_\tau) \sim 0.3$

$$\Gamma(\tau \to \nu_\tau d \bar{u}) = 3 \, \frac{G_F^2 m_\tau^5}{192\pi^3} \left(1 + \frac{\alpha_s(m_\tau)}{\pi}\right)$$

$$= 13.4 \times 10^{-10} \text{ MeV}$$

c) then

$$\Gamma(\tau) = (\text{sum of the three partial widths above})$$

$$= 21.5 \times 10^{-10}\ \text{MeV}$$

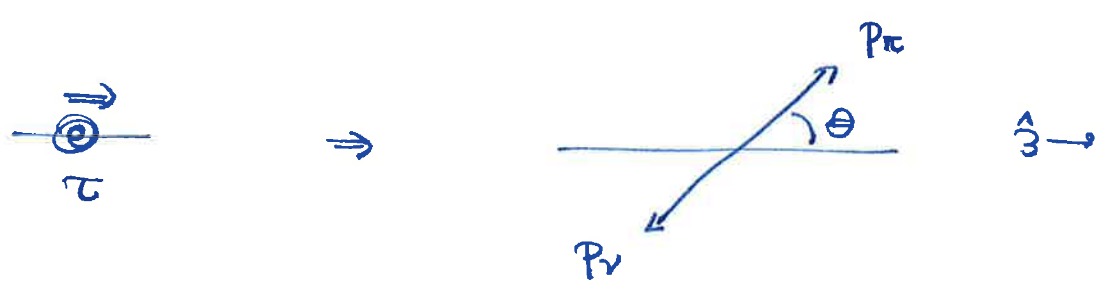
$$\tau = \frac{\hbar}{\Gamma(\tau)} = \frac{6.582 \times 10^{-22}\ \text{MeV}\cdot\text{sec}}{21.5 \times 10^{-10}\ \text{MeV}} = 3.06 \times 10^{-13}\ \text{sec}$$

$$BR(\tau \to \nu_\tau e \bar{\nu}_e) = \frac{4.05}{21.5} = 18.8\%$$

the PDG values are: $\tau = 2.90 \times 10^{-13}$ sec

$BR = 17.8\%$

d.)



$$P_\tau = (m_\tau, 0, 0, 0)$$

$$P_\pi = (\mathcal{E}, p\sin\theta, 0, p\cos\theta)$$

$$P_\nu = (p, -p\sin\theta, 0, -p\cos\theta)$$

$$\mathcal{E} = \frac{m_\tau^2 + m_\pi^2}{2m_\tau} \qquad p = \frac{m_\tau^2 - m_\pi^2}{2m_\tau}$$

the spinors are

$$u(\tau) = \sqrt{m_\tau}\begin{pmatrix} 1 \\ 0 \\ \hline 1 \\ 0 \end{pmatrix} \qquad u(\nu_\tau) = \begin{pmatrix} \eta \\ 0 \\ 0 \end{pmatrix}$$

where η = rotation of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ by $-(\pi-\theta)$

$$= \begin{pmatrix} -\cos\theta/2 \\ -\sin\theta/2 \end{pmatrix}$$

e.)

$$\mathcal{M} = \frac{4G_F}{\sqrt{2}} \langle \nu_\tau \pi^- | (\nu_\tau^\dagger \bar{\sigma}^\mu \tau)(d^\dagger \bar{\sigma}_\mu u) | \tau \rangle$$

$$= \frac{4G_F}{\sqrt{2}} \langle \nu_\tau | \nu_{\tau L}^\dagger \bar{\sigma}^\mu \tau_L | \tau \rangle \langle \pi^-(p) | (d_L^\dagger \bar{\sigma}_\mu u_L) | 0 \rangle$$

for the hadronic matrix element

$$d_L^\dagger \bar{\sigma}^\mu u_L = \bar{Q} \gamma^\mu \left(\frac{1-\gamma^5}{2}\right) \sigma^- Q \qquad \sigma^- = \frac{\sigma^1 - i\sigma^2}{2}$$

$$= \frac{1}{2} \bar{Q} \gamma^\mu \frac{\sigma^1 - i\sigma^2}{2} Q - \frac{1}{2} \bar{Q} \gamma^\mu \gamma^5 \left(\frac{\sigma^1 - i\sigma^2}{2}\right) Q$$

$$= (\text{vector}) - \frac{1}{2}\underline{\left(j^{\mu 51} - i j^{\mu 52}\right)}$$

by $\underline{\text{parity}}$ only the $j^{\mu 5}$ terms can create 1 pion

$$|\pi^-\rangle = \frac{1}{\sqrt{2}}\left(|\pi^1\rangle - i|\pi^2\rangle\right)$$

so $\langle \pi^- | = \frac{1}{\sqrt{2}}\left(\langle \pi^1 | + i \langle \pi^2 |\right)$

in all

$$\langle \pi^-(p) | d_L^\dagger \bar{\sigma}^\mu u_L | 0 \rangle$$

$$= -\frac{1}{2}\frac{1}{\sqrt{2}} \left(\langle \pi^1 | + i \langle \pi^2 | \right) \left(j^{\mu 5 1} - i j^{\mu 5 2} \right) | 0 \rangle$$

$$= -\frac{1}{2}\frac{1}{\sqrt{2}} \cdot 2 \cdot \left[-i f_\pi p^\mu \right] = i \frac{f_\pi p^\mu}{\sqrt{2}}$$

the leptonic part of the matrix element is

$$\langle \nu_\tau | \nu_\tau^\dagger \bar{\sigma}^\mu \tau | \tau \rangle = \sqrt{m_\tau \cdot 2 p_\nu} \quad \eta^\dagger \bar{\sigma}^\mu \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \sqrt{m_\tau \cdot 2 p_\nu} \cdot \left(-\cos\theta/2 \quad -\sin\theta/2 \right) \left(1, -\sigma^1, -\sigma^2, -\sigma^3 \right)^\mu \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \sqrt{2 m_\tau p_\nu} \left(-\cos\theta/2 \, , \sin\theta/2 \, , -i \sin\theta/2 \quad \cos\theta/2 \right)^\mu$$

$$\mathcal{M} = \frac{4 G_F}{\sqrt{2}} \frac{i f_\pi}{\sqrt{2}} \sqrt{2 m_\tau p_\nu} \cdot \left[- \mathcal{E} \cos\theta/2 - p(\sin\theta/2 \sin\theta + \cos\theta/2 \cos\theta) \right]$$

$$= -i \frac{4 G_F}{2} f_\pi \sqrt{2 m_\tau p} \left[\mathcal{E} + p \right] \cos\theta/2$$

$$2 m_\tau p = m_\tau^2 - m_\pi^2 \qquad \mathcal{E} + p = m_\tau$$

$$= -i \, 2 G_F f_\pi m_\tau (m_\tau^2 - m_\pi^2)^{1/2} \cos\theta/2$$

f.) $$\Gamma(\tau \to \pi \nu_\tau) = \frac{1}{2 m_\tau} \frac{1}{8\pi} \int_{-1}^{1} \frac{d\cos\theta}{2} \cdot \left(\frac{2p}{m_\tau} \right)$$

$$\cdot \left(2 G_F f_\pi m_\tau \right)^2 (m_\tau^2 - m_\pi^2) \cos^2\theta/2$$

$$\Gamma(\tau \to \pi \nu_\tau) = \frac{G_F^2 f_\pi^2 m_\tau^3}{4\pi} \cdot \left(1 - \frac{m_\pi^2}{m_\tau^2}\right)^2 \cdot \underbrace{\int_{-1}^{1} \frac{d\cos\theta}{2} \cos^2\theta/2}_{= 1/2}$$

$$\Gamma(\tau \to \pi \nu_\tau) = \frac{G_F^2 f_\pi^2 m_\tau^3}{8\pi} \left(1 - \frac{m_\pi^2}{m_\tau^2}\right)^2$$

$$= 2.6 \times 10^{-13}\ \mathrm{GeV} = 2.6 \times 10^{-10}\ \mathrm{MeV}$$

so

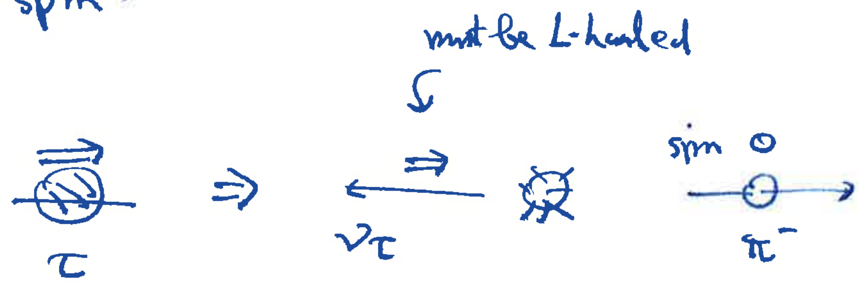
$$BR(\tau \to \pi \nu_\tau) = \frac{2.6}{21.5} = 12\%$$

the PDG value is $\underline{11.5\%}$

f.)

$$\frac{d\Gamma}{d\cos\theta} \sim \frac{\cos^2\theta/2}{2} \sim (1 + \cos\theta)$$

so, generally, the π^- is emitted in the direction of the τ^- spin!



2.) a.)

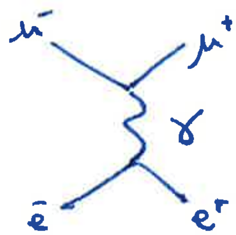
$e_L^- e_R^+ \to \mu_L^- \mu_R^+ :\qquad \frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{2s}(1+\cos\theta)^2$

$e_L^- e_R^+ \to \mu_R^- \mu_L^+ :\qquad \frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{2s}(1-\cos\theta)^2$

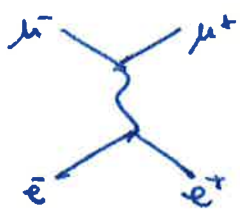
$e_R^- e_L^+ \to \mu_L^- \mu_R^+ :\qquad \frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{2s}(1-\cos\theta)^2$

$e_R^- e_L^+ \to \mu_R^- \mu_L^+ :\qquad \frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{2s}(1+\cos\theta)^2$

b.)



$$= (-e)^2 (u^\dagger \bar\sigma^\mu v)_\mu \frac{1}{s} (v^\dagger \bar\sigma_\mu u)_e$$



$$= \left(\frac{e}{c_w s_w} Q_Z\right)_e \left(\frac{e}{c_w s_w}\right)_\mu (u^\dagger \bar\sigma v)_\mu \frac{1}{s-m_Z^2} (v^\dagger \bar\sigma_\mu u)_e$$

the matrix element $\mathcal{M}$ is the sum of these quantum mechanical amplitudes so.

$$\mathcal{M}_{\text{Electroweak}} = \mathcal{M}_{QED}\left[1 + \frac{1}{c_w s_w}(Q_Z)_e (Q_Z)_\mu \cdot \frac{s}{s-m_Z^2}\right]$$

and

$$\left.\frac{d\sigma}{d\cos\theta}\right|_{\text{electroweak}} = \left.\frac{d\sigma}{d\cos\theta}\right|_{QED} \left| 1 + \frac{(Q_Z)_e (Q_Z)_\mu}{c_w s_w} \frac{s}{s-m_Z^2}\right|^2$$

now $\quad Q_Z = \begin{cases} -\frac{1}{2} + s_w^2 & e^-_L \ \mu^-_L \\ s_w^2 & e^-_R \ \mu^-_R \end{cases}$

so

$$\frac{d\sigma}{d\cos\theta}(e^-_L e^+_R \to \mu^-_L \mu^+_R)\Big|_{EW} = \frac{d\sigma}{d\cos\theta}(e^-_L e^+_R \to \mu^-_L \mu^+_R)\Big|_{QED}$$

$$\cdot \left| 1 + \frac{(\frac{1}{2} - s_w^2)^2}{c_w^2 s_w^2} \frac{s}{s - m_Z^2} \right|^2$$

c.) The analysis for the other states is similar. The factors are

$e^-_L e^+_R \to \mu^-_R \mu^+_L$: $\quad \left| 1 - \frac{(\frac{1}{2} - s_w^2) s_w^2}{c_w^2 s_w^2} \frac{s}{s - m_Z^2} \right|^2$

$e^-_R e^+_L \to \mu^-_L \mu^+_R$ $\quad \left| 1 - \frac{s_w^2 (\frac{1}{2} - s_w^2)}{c_w^2 s_w^2} \frac{s}{s - m_Z^2} \right|^2$

$e^-_R e^-_L \to \mu^-_R \mu^+_L$ $\quad \left| 1 + \frac{s_w^4}{c_w^2 s_w^2} \frac{s}{s - m_Z^2} \right|^2$

d.) For a process with $\frac{d\sigma}{d\cos\theta} \sim (1 + \cos\theta)^2$

$$\int_{-1}^{1} d\cos\theta \, (1 + \cos\theta)^2 = \frac{8}{3}$$

$$\int_{0}^{1} d\cos\theta \, (1 + \cos\theta)^2 = \frac{7}{3}$$

$$\int_{-1}^{0} d\cos\theta \, (1 + \cos\theta)^2 = \frac{1}{3}$$

so $$A_{FB} = \frac{7-1}{8} = \frac{3}{4}$$

e.) For $e^-_L e^+_R \to \mu^-_L \mu^+_R$, the forward cross section has integral 7 and the backward cross section has integral 1. Similarly for $e^-_R e^+_L \to \mu^-_R \mu^+_L$

so A_{FB} for the unpolarized process (sum of 4 cross sections) is

$$A_{FB} = \frac{3\left[F_{LL}(s) + F_{RR}(s) - F_{LR}(s) - F_{RL}(s)\right]}{4\left[F_{LL}(s) + F_{RR}(s) + F_{LR}(s) + F_{RL}(s)\right]}$$

where the F's are the factors above

$$F_{LL} = \left|1 + \frac{(\frac{1}{2} - s_W^2)^2}{c_W^2 s_W^2} \frac{s}{s - m_Z^2}\right|^2$$

$$F_{RR} = \left|1 + \frac{s_W^4}{c_W^2 s_W^2} \frac{s}{s - m_Z^2}\right|^2$$

$$F_{LR} = F_{RL} = \left|1 - \frac{s_W^2(\frac{1}{2} - s_W^2)}{c_W^2 s_W^2} \frac{s}{s - m_Z^2}\right|^2$$

for $s \ll m_Z$ $\quad \frac{s}{s - m_Z^2} \approx \frac{-s}{m_Z^2} \to 0$

and $A_{FB} \to 0$

just on the Z resonance, the $\left(\frac{s}{s - m_Z^2 + m_Z \Gamma_Z}\right)$ terms are dominant. Then

$$A_{FB} = \frac{3}{4}\,\frac{\left(\left(\tfrac{1}{2}-s_W^2\right)^2\right)^2 + \left(s_W^4\right)^2 - 2\left(s_W^2\left(\tfrac{1}{2}-s_W^2\right)\right)^2}{\left(\left(\tfrac{1}{2}-s_W^2\right)^2\right)^2 + \left(s_W^4\right)^2 + 2\left(s_W^2\left(\tfrac{1}{2}-s_W^2\right)\right)^2}$$

$$= \frac{3}{4}\,\frac{\left[\left(\tfrac{1}{2}-s_W^2\right)^2 - s_W^4\right]^2}{\left[\left(\tfrac{1}{2}-s_W^2\right)^2 + s_W^4\right]^2}$$

$$= \frac{3}{4}\,\frac{\tfrac{1}{4} - s_W^2}{\left[\tfrac{1}{4} - s_W^2 + 2s_W^4\right]^2}$$

$$A_{FB} \simeq \frac{3}{4}\,\frac{\left(1-4s_W^2\right)^2}{\left[1-4s_W^2+8s_W^4\right]^2}$$

f.) As $s/m_Z^2 \gg 1$

$$\frac{d\sigma}{d\cos\theta}\left(e_L^- e_R^+ \to \mu_L^- \mu_R^+\right) \to \frac{\pi\alpha^2}{2s}\left|1 + \frac{\left(\tfrac{1}{2}-s_W^2\right)^2}{c_W^2 s_W^2}\right|^2 (1+\cos\theta)^2$$

This simplifies

$$1 + \frac{\left(\tfrac{1}{2}-s_W^2\right)^2}{c_W^2 s_W^2} = \frac{s_W^2(1-s_W^2) + \tfrac{1}{4} - s_W^2 + s_W^4}{c_W^2 s_W^2}$$

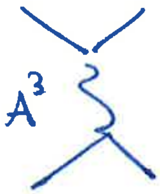
$$= \frac{1/4}{c_W^2 s_W^2} = \frac{1}{4s_W^2} + \frac{1}{4c_W^2}$$

Since $c_W^2 + s_W^2 = 1$

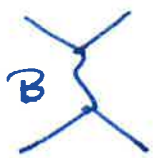
Then $$\frac{d\sigma}{d\cos\Theta}(e_L^- e_R^+ \to \mu_L^- \mu_R^+) \to \frac{\pi\alpha^2}{2s}\left|\left(\frac{1}{2s_w}\right)^2 + \left(\frac{1}{2c_w}\right)^2\right|^2$$

g.) In terms of A^3 and B, e_L^- and μ_L^- have

$I^3 = Y = -\frac{1}{2}$ so



$$= \left(-\tfrac{1}{2}g\right)^2 \frac{1}{s}(u^\dagger\bar\sigma v)(v^\dagger\bar\sigma u)$$



$$= \left(-\tfrac{1}{2}g'\right)^2 \frac{1}{s}(u^\dagger\sigma v)(v^\dagger\sigma u)$$

but $$g = \frac{e}{s_w} \qquad g' = \frac{e}{c_w}$$

so $$\mathcal{M} = e\,\frac{1}{s}(u^\dagger\sigma v)(v^\dagger\sigma u)\cdot\left[\left(\frac{1}{2s_w}\right)^2 + \left(\frac{1}{2c_w}\right)^2\right]$$

$$\frac{d\sigma}{d\cos\Theta} = \frac{\pi\alpha^2}{2s}(1+\cos\Theta)^2\left[\left(\frac{1}{2s_w}\right)^2 + \left(\frac{1}{2c_w}\right)^2\right]$$

∴ agreement with (f)

for the other helicity states:

$e^-_L e^+_R \to \mu^-_R \mu^+_L$

γ, Z: $$F_{LR} \to \left| 1 - \frac{s_W^2(\frac{1}{2} - s_W^2)}{c_W^2 s_W^2} \right|^2$$

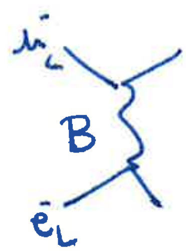
$$= \left| \frac{s_W^2(1-s_W^2) - \frac{1}{2}s_W^2 + s_W^4}{c_W^2 s_W^2} \right|^2$$

$$= \left| \frac{\frac{1}{2}s_W^2}{c_W^2 s_W^2} \right|^2 = \left| \frac{1}{2c_W^2} \right|^2$$

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{2s}(1-\cos\theta)^2 \left| \frac{1}{2c_W^2} \right|^2$$

A^3, B : A^3 does not couple to μ^-_R so we have

only



$(-\frac{1}{2}g')(-g')$ μ^-_R has $I=0$ $Y=-1$

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{2s}(1-\cos\theta)^2 \left| \left(-\frac{1}{2c_W}\right)\left(-\frac{1}{c_W}\right) \right|^2$$

For $e^-_R e^+_L \to \mu^-_L \mu^+_R$, the analysis is the same

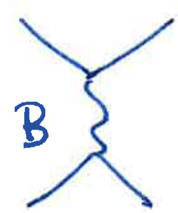
for $e^-_R e^+_L \to \mu^-_R \mu^+_L$

γ, Z $\qquad F_{RR} \longrightarrow \left| 1 + \frac{s_W^4}{c_W^2 s_W^2} \right|^2$

$$= \left| 1 + \frac{s_W^2}{c_W^2} \right|^2 = \left| \frac{1}{c_W^2} \right|$$

so $$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{2s} \left(\frac{1}{c_W^2} \right)^2 (1+\cos\theta)^2$$

$A^3, B.$ $\quad A^3$ doesn't couple, B gives



$$\sim (-g')^2 = \left(-\frac{e}{c_W} \right)^2$$

so $$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{2s} (1+\cos\theta)^2 \left| \frac{1}{c_W^2} \right|^2$$