# Physics 152/252

## Problem Set #7 — Solutions

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1.) a.)

$$S = \int d^4x \left[ \tfrac{1}{2}(\partial_\mu \phi_i)^2 - V(\phi) \right]$$

$$\delta S = \int d^4x \left[ \partial_\mu \delta\phi_i \, \partial^\mu \phi_i - \delta\phi_i \frac{\partial}{\partial \phi_i} V \right]$$

$$= \int d^4x \, \delta\phi_i \left[ -\partial_\mu \partial^\mu \phi_i - \frac{\partial}{\partial \phi_i} V \right]$$

$$\delta S = 0 \Rightarrow [\;] = 0 \Rightarrow \qquad \partial_\mu \partial^\mu \phi_i + \frac{\partial}{\partial \phi_i} V = 0$$

b.)

$$\phi_i = \Phi_i + \eta_i(x) \qquad \text{where} \qquad \frac{\partial}{\partial \phi_i} V \Big|_{\phi = \Phi} = 0$$

expanding the equation of motion

$$\partial_\mu \partial^\mu (\Phi_i + \eta_i(x)) + \frac{\partial}{\partial \phi_i} V(\Phi_i + \eta_i(x)) = 0$$

$$\partial_\mu \partial^\mu \eta_i(x) + \frac{\partial V}{\partial \phi_i}\Big|_\Phi + \eta_j(x) \frac{\partial^2 V}{\partial \phi_i \partial \phi_j}\Big|_\Phi + \cdots = 0$$

then

$$\partial_\mu \partial^\mu \eta_i + M^2_{ij} \eta_j + O(\eta^2) = 0$$

where $$M^2_{ij} = \frac{\partial^2 V}{\partial\phi_i \partial\phi_j}\Big|_{\phi=\Phi}$$

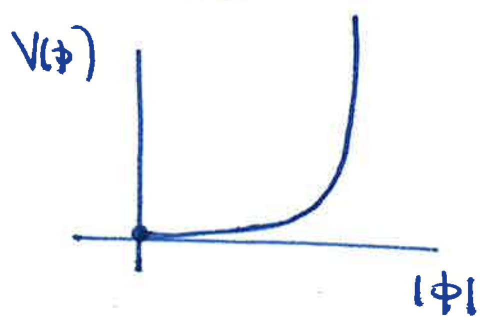
If we diagonalize $M_{ij}$ $$\eta_i(x) = \sum_a \xi_i^a \, \theta^a(x)$$

with $$M_{ij}\,\xi_j^a = \lambda^a \xi_j^a$$

then $$\partial_\mu \partial^\mu \theta^a(x) + \lambda^a \theta^a(x) = 0$$

so $$\lambda^a = \text{eigenvalue of } M^2 = m^2 \text{ of the field } \theta^a(x).$$

c.) $V(\phi)$ is spherically symmetric in $\phi_i$

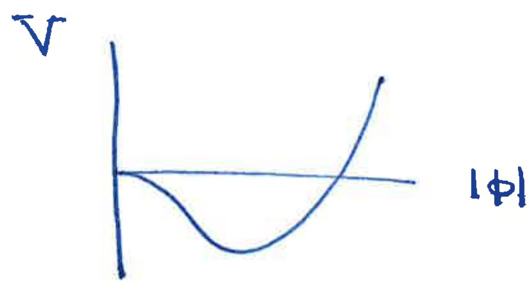



The minimum of $V$ is at $\phi_0 = 0$

$$M^2_{ij} = \frac{\partial^2}{\partial\phi_i \partial\phi_j} V\Big|_0 = \mu^2 \delta_{ij}$$

so all $n$ particles have mass $= \mu$

d.) The potential is spherically symmetric with shape




The minima occur at vectors $\phi_i$ s.t. $|\phi| = v$

One example is $\phi = (0\ 0 \cdot\ 0, v)$

To find the minimum

$$0 = \frac{\partial V}{\partial v} = \frac{\partial}{\partial v}\left(-\tfrac{1}{2}\mu^2 v^2 + \tfrac{1}{4}\lambda v^4\right)$$

$$= -\mu^2 v + \lambda v^3$$

so $v = \mu/\sqrt{\lambda}$

e.) $$M^2_{ij} = \frac{\partial^2}{\partial\phi_i\partial\phi_j} V(\phi) \Big|_{\phi = (0\,0\,..\,0\,v)}$$

$$= \frac{\partial}{\partial\phi_i}\left[-\mu^2\phi_j + \lambda\phi^2\phi_j\right]$$

$$= -\mu^2\delta_{ij} + \lambda\phi^2\delta_{ij} + 2\lambda\phi_i\phi_j$$

at $\phi^2 = v^2 = \frac{\mu^2}{\lambda}$ $$= 0 + 2\lambda\phi_i\phi_j$$

The matrix $M^2$ is then

$$M^2 = \begin{pmatrix} 0 & & & & \\ & 0 & & 0 & \\ & & 0 & & \\ & & & 0 & \\ & 0 & & & 2\lambda v^2 \end{pmatrix}$$

with $2\lambda v^2 = 2\mu^2$

so $(n-1)$ masses are $0$ (Goldstone bosons)

$1$ mass is $\sqrt{2}\mu$

2.) a.) For $Q > m_b$, $b_0 = 11 - \frac{2}{3} n_f$ with $n_f = 5$:

$$b_0 = 11 - \frac{10}{3} = \frac{23}{3}$$

then

$$\alpha_s(Q) = \frac{\alpha_s(Q_0)}{1 + \frac{23}{3} \frac{\alpha_s(Q_0)}{2\pi} \log(Q/Q_0)}$$

then

$$\alpha_s(Q_0) = 0.118 \text{ at } Q_0 = 91.$$

$$\Rightarrow \alpha_s(m_b) = 0.212$$

For $m_b > Q > m_c$, $b_0 = \frac{25}{3}$

$$\alpha_s(Q) = \frac{\alpha_s(m_b)}{1 + \frac{25}{3} \frac{\alpha_s(m_b)}{2\pi} \log(Q/m_b)}$$

this gives

$$\alpha_s(2.) = 0.268$$
$$\alpha_s(1.28) = 0.318$$

b.)

$$\frac{d}{d\log Q} m_f(Q) = -8 \frac{\alpha_s(Q)}{4\pi} m_f(Q)$$

$$\frac{dm_f}{m_f} = -\frac{8}{4\pi} \frac{\alpha_s(Q_0)}{1 + b_0 \frac{\alpha_s(Q_0)}{2\pi} \log Q/Q_0} \, d\log Q$$

$$d \log m_f = -\frac{8}{4\pi}\frac{2\pi}{b_0}\, d\log\left(1 + b_0 \frac{\alpha_s(Q_0)}{2\pi}\log Q/Q_0\right)$$

$$= -\frac{4}{b_0}\, d\log\frac{1}{\alpha_s(Q)} = \frac{4}{b_0}\log\alpha_s(Q)$$

so

$$\frac{m_f(Q)}{m_f(Q_0)} = \left[\frac{\alpha_s(Q_0)}{\alpha_s(Q)}\right]^{4/b_0}$$

c.) Then

$$m_c(2) = m_c(1.28)\cdot\left[\frac{\alpha_s(2)}{\alpha_s(1.28)}\right]^{12/25}$$

$$= m_c(1.28)\cdot 0.921$$

so at 2 GeV, we have:

| $m_u$ | $m_d$ | $m_s$ | $m_c$ |
|---|---|---|---|
| 0.0023 | 0.0048 | 0.095 | 1.179 |

GeV

d.) To move these values to $Q = m_b$, we multiply by

$$\left[\frac{\alpha_s(4.18)}{\alpha_s(2)}\right]^{12/25} = 0.894$$

Then, at $Q = m_b$:

| $m_u$ | $m_d$ | $m_s$ | $m_c$ | $m_b$ |
|---|---|---|---|---|
| .00206 | .00429 | .0850 | 1.05 | 4.18 |

GeV

e.) To move these values to $Q = m_t = 164.$ we first need to find

$$\alpha_s(164.) = \frac{\alpha_s(91.)}{1 + \frac{23}{3}\frac{1}{2\pi}\alpha_s(91.)\log\frac{164}{91.}} = 0.109$$

Then we need to multiply the above by

$$\left[\frac{\alpha_s(164.)}{\alpha_s(4.18)}\right]^{12/23} = 0.707$$

Then $Q = m_t$.

| $m_u$ | $m_d$ | $m_s$ | $m_c$ | $m_b$ | $m_t$ |
|---|---|---|---|---|---|
| .0015 | .0030 | 0.060 | 0.742 | 2.95 | 164. |

GeV

$m_q/m_t$ =

| | | | | | |
|---|---|---|---|---|---|
| $0.88\times10^{-5}$ | $1.8\times10^{-5}$ | $3.7\times10^{-4}$ | $4.5\times10^{-3}$ | $1.8\times10^{-2}$ | 1 |