Physics 152/252

Problem Set #6 – Solutions

1.) a) We found earlier in the course

$$\sigma(e^+e^- \to q_f \bar{q}_f) = \frac{4}{3}\frac{\pi\alpha^2}{s} \cdot Q_f^2 \cdot 3$$

The factor 3 appears because the quarks may be produced with any of 3 colors. For the reverse process

$$q_f \bar{q}_f \to e^+e^- \text{ or } \mu^+\mu^-$$

The colors of the quark and antiquark must match. The probability of this is $\frac{1}{3}$. This is the only change. So

$$\sigma(q_f \bar{q}_f \to \mu^+\mu^-) = \frac{4}{3}\frac{\pi\alpha^2}{s} \cdot Q_f^2 \cdot \frac{1}{3}$$

b.) $$\hat{s} = [M(\mu^+\mu^-)]^2 = (p_1 + p_2)^2$$

$$= (x_1 P_1 + x_2 P_2)^2 = 2x_1 x_2 P_1 \cdot P_2$$

if we can set $P_1^2 = P_2^2 \approx 0$

$$\hat{s} = x_1 x_2 \cdot s = M^2$$

c) $\quad P_1 = (E, 0, 0, E) \qquad P_2 = (E, 0, 0, -E)$

$$s = 4E^2$$

$$p_1 = (x_1 E, 0, 0, x_1 E) \qquad p_2 = (x_2 E, 0, 0, -x_2 E)$$

$$p_1 + p_2 = (\mathcal{E}, 0, 0, \mathcal{P})$$

with $\quad \mathcal{E} = (x_1 + x_2) E \qquad \mathcal{P} = (x_1 - x_2) E$

then $\quad \mathcal{P}/\mathcal{E} = \tanh y = \dfrac{x_1 - x_2}{x_1 + x_2}$

$$\mathcal{E}^2 - \mathcal{P}^2 = \left[(x_1 + x_2)^2 - (x_1 - x_2)^2\right] E^2$$

$$= 4 x_1 x_2 E^2 = x_1 x_2 s = M^2$$

so $\quad \mathcal{E} = M \cosh y \qquad \mathcal{P} = M \sinh y$

then $\quad \cosh y = \dfrac{\mathcal{E}}{M} = \dfrac{x_1 + x_2}{2\sqrt{x_1 x_2}} = \frac{1}{2}(e^y + e^{-y})$

$\quad \sinh y = \dfrac{\mathcal{P}}{M} = \dfrac{x_1 - x_2}{2\sqrt{x_1 x_2}} = \frac{1}{2}(e^y - e^{-y})$

then $\quad e^y = \sqrt{\dfrac{x_1}{x_2}} \qquad e^{-y} = \sqrt{\dfrac{x_2}{x_1}}$

$$y = \frac{1}{2} \log\left(\frac{x_1}{x_2}\right)$$

d.) conversely $\quad e^{2y} = \frac{x_1}{x_2} \qquad \frac{M^2}{s} = x_1 x_2$

so

$$x_1 = \frac{M}{\sqrt{s}} e^{y} \qquad x_2 = \frac{M}{\sqrt{s}} e^{-y}$$

e.) $$\frac{\partial(x_1, x_2)}{\partial(M, y)} = \begin{vmatrix} \frac{1}{\sqrt{s}} e^{y} & \frac{1}{\sqrt{s}} e^{-y} \\ \frac{M}{\sqrt{s}} e^{y} & -\frac{M}{\sqrt{s}} e^{-y} \end{vmatrix} = \frac{2M}{s}$$

so

$$\sigma = \sum_f \int dx_1\, dx_2 \left(f_f(x_1) f_{\bar{f}}(x_2) + (f \leftrightarrow \bar{f}) \right) \sigma(q_f \bar{q}_f \to \mu^+ \mu^-)$$

$$= \sum_f \int dM\, dy \cdot \frac{2M}{s} \left(f_f(x_1) f_{\bar{f}}(x_2) + (f \leftrightarrow \bar{f}) \right) \cdot \left(\frac{4}{9} Q_f^2 \frac{\pi \alpha^2}{M^2} \right)$$

$$\frac{d\sigma}{dM\, dy} = \sum_f \frac{2M}{s} \left(f_f(x_1) f_{\bar{f}}(x_2) + (f \leftrightarrow \bar{f}) \right) \cdot \left(\frac{4}{9} Q_f^2 \frac{\pi \alpha^2}{M^2} \right)$$

$$M^2 = x_1 x_2 s$$

$$\frac{d\sigma}{dM\, dy} = \sum_f Q_f^2 \left[x_1 f_f(x_1)\; x_2 f_{\bar{f}}(x_2) + (f \leftrightarrow \bar{f}) \right] \cdot \left(\frac{8}{9} \frac{\pi \alpha^2}{M^3} \right)$$

f.) At the LHC, for $M = 90$ GeV,

$$M^2/s = x_1 x_2 = 5\times10^{-5} = (7\times10^{-3})^2$$

Near $y = 0$ $\quad x_1 \approx x_2 \approx 10^{-2}$; both x_1 and x_2 are in the sea region

For a valence quark to be involved $x_1 \sim 0.1$

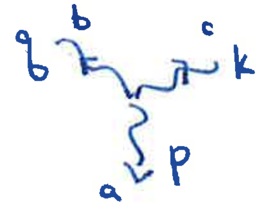
$$x_2 \doteq 5\times10^{-4}$$

$$y = \frac{1}{2}\log\left(\frac{x_1}{x_2}\right) \sim 2.5$$

So, for $|y| < 2.5$ $\quad pp \to \mu^+\mu^-$ is mostly sea-sea

$|y| > 2.5$ $\quad pp \to \mu^+\mu^-$ is mostly sea-valence.

2) a.) Write (4) more symmetrically



the right-hand side of (4) becomes.

$$g_s f^{abc}\left[(k-p)\cdot\varepsilon_q^*\ \varepsilon_p^*\cdot\varepsilon_k^* + (p-q)\cdot\varepsilon_k^*\ \varepsilon_q^*\cdot\varepsilon_p^* + (q-k)\cdot\varepsilon_p^*\ \varepsilon_q^*\cdot\varepsilon_k^*\right]$$

The term in brackets is manifestly symmetric under the cycling $k\to p\to q\to k$.

Interchange $\qquad k\leftrightarrow p \qquad \varepsilon_k^*\leftrightarrow\varepsilon_p^*$

the term in brackets gets a (-1). But also $a\leftrightarrow c$

changes
$$f^{abc} = f^{cba} = -f^{abc}$$

so this expression is completely symmetric under

$$k\leftrightarrow p \qquad \varepsilon_k^*\leftrightarrow\varepsilon_p^* \qquad a\leftrightarrow c$$

or any other interchanges.

b.)
$$p = \left(E,\ 0\ \ 0\ \ E - \frac{q_T^2}{2z(1-z)E}\right)$$
$$q = \left(zE,\ q_T,\ 0,\ zE - \frac{q_T^2}{2zE}\right)$$
$$k = \left((1-z)E,\ -q_T,\ 0,\ (1-z)E - \frac{q_T^2}{2(1-z)E}\right)$$

and $p^2 = q_T^2/(z(1-z))$ as in the example of $g \to gg$

c.) for p:

$$\varepsilon_{pR} = \tfrac{1}{\sqrt{2}}(0\ 1\ i\ 0)$$
$$\varepsilon_{pL} = \tfrac{1}{\sqrt{2}}(0\ 1\ -i\ 0)$$

for q:

$$\varepsilon^*_{qR} = \tfrac{1}{\sqrt{2}}\left(0, 1, -i, -\frac{q_T}{zE}\right)$$
$$\varepsilon^*_{qL} = \tfrac{1}{\sqrt{2}}\left(0, 1, i, -\frac{q_T}{zE}\right)$$

rotate so $\vec{\varepsilon}^*_q \cdot \vec{q} = 0$

for k:

$$\varepsilon^*_{kR} = \tfrac{1}{\sqrt{2}}\left(0, 1, -i, +\frac{q_T}{(1-z)E}\right)$$
$$\varepsilon^*_{kL} = \tfrac{1}{\sqrt{2}}\left(0\ 1, i, +\frac{q_T}{(1-z)E}\right)$$

Make a table of vector products $\varepsilon \cdot p$ ...

	p	q	k
$\varepsilon_{pR,L}$:	0	$-\frac{q_T}{\sqrt{2}}$	$+\frac{q_T}{\sqrt{2}}$
$\varepsilon_{qR,L}$:	$+\frac{1}{\sqrt{2}}\frac{q_T}{z}$	0	$\frac{1}{\sqrt{2}}\left(q_T + \frac{(1-z)}{z}q_T\right) = \frac{1}{\sqrt{2}}\frac{q_T}{z}$
$\varepsilon_{kR,L}$:	$-\frac{1}{\sqrt{2}}\frac{q_T}{(1-z)}$	$-\frac{1}{\sqrt{2}}\frac{q_T}{(1-z)}$	0

$$\frac{1}{\sqrt{2}}\left(-q_T - q_T\frac{z}{(1-z)}\right)$$

Make a table of vector products $\varepsilon \cdot \varepsilon$

	ε_{pR}	ε_{pL}	ε^*_{qR}	ε^*_{qL}	ε^*_{kR}	ε^*_{kL}
ε_{pR}	0	-1	-1	0	-1	0
ε_{pL}	-1	0	0	-1	0	-1
ε^*_{qR}			0	-1	0	-1
ε^*_{qL}			-1	0	-1	0
ε^*_{kR}					0	-1
ε^*_{kL}					-1	0

d.) For $g_R \to g_L g_L$

$$\mathcal{M} = g_s f^{abc} \Big[(\ldots)\, \underbrace{\varepsilon_{pR} \cdot \varepsilon^*_{kL}}_{=0} - (\ldots)\, \underbrace{\varepsilon_{pR} \cdot \varepsilon^*_{qL}}_{0}$$

$$+ (\ldots)\, \underbrace{\varepsilon^*_{qL} \cdot \varepsilon^*_{kL}}_{0} \Big]$$

$$= 0 \quad !$$

e.) for $g_R \to g_R\, g_L$

$$\mathcal{M} = g_s f^{abc}\Big[(\cdots)\,\underbrace{\varepsilon_p\cdot\varepsilon^*_{kL}}_{0} - (p\cdot\varepsilon^*_{kL} + q\cdot\varepsilon^*_{kL})\,\overbrace{\varepsilon_{pR}\cdot\varepsilon^*_{qR}}^{-1}$$

$$+ (q\cdot\varepsilon_{pR} - k\cdot\varepsilon_{pR})\,\underbrace{\varepsilon^*_{qR}\cdot\varepsilon^*_{kL}}_{-1}\Big]$$

$$= g_s f^{abc}\Big[0 + \Big[-\frac{1}{\sqrt2}\frac{q_T}{(1-z)} - \frac{1}{\sqrt2}\frac{q_T}{(1-z)}\Big]$$

$$- \Big[+ \Big(-\frac{q_T}{\sqrt2}\Big) \cdot \frac{q_T}{\sqrt2}\Big]\Big]$$

$$= g_s f^{abc}\Big[-\frac{q_T}{\sqrt2}\cdot 2\cdot\Big(\frac{1}{1-z} - 1\Big)\Big]$$

$$= -g_s f^{abc}\sqrt2\, q_T\,\frac{z}{1-z}$$

for $g_R \to g_L\, g_R$ just interchange bec $z \leftrightarrow 1-z$

$$\mathcal{M} = g_s f^{abc}\sqrt2\, g_T\left(\frac{1-z}{z}\right)$$

f.) for $g_R \to g_R\, g_R$

$$\mathcal{M} = g_s f^{abc}\Big[(k\cdot\varepsilon^*_q + p\cdot\varepsilon^*_q)\,\overbrace{\varepsilon_{pR}\cdot\varepsilon^*_{kR}}^{-1}$$

$$- (p\cdot\varepsilon^*_k + q\cdot\varepsilon^*_k)\,\underbrace{\varepsilon_{pR}\cdot\varepsilon^*_{qR}}_{-1} + (\cdot\cdot)\,\underbrace{\varepsilon^*_{qR}\cdot\varepsilon^*_{kR}}_{0}\Big]$$

$$= g_s f^{abc} \left[\frac{1}{\sqrt{2}} \frac{q_T}{z} \cdot 2 \cdot (-1) - \left(-\frac{1}{\sqrt{2}} \frac{q_T}{(1-z)} \right) \cdot 2 \cdot (-1) \right]$$

$$= -g_s f^{abc} \sqrt{2}\, q_T \left(\frac{1}{z} + \frac{1}{(1-z)} \right) = -g f^{abc} \sqrt{2} \frac{q_T}{z(1-z)}$$

g.) By parity $\mathcal{M}(g_L \to g_R g_R) = \mathcal{M}(g_R \to g_L g_L)$

etc

h.) Sum over colors: $\sum_{abc} f^{abc} f^{abc} = \sum_a N \delta^{aa}$

$$= N(N^2-1) \quad \text{for } SU(N)$$

for SU(3)

$$\frac{1}{8} \sum_{abc} f^{abc} f^{abc} = 3$$

For SU(2), the claimed identity is

$$\varepsilon^{abc} \varepsilon^{dbc} = \delta^{ad} \delta^{bb} - \delta^{ab} \delta^{bd} = 3\delta^{ad} - \delta^{ad} = 2\delta^{ad}$$

i.) We need

$$\underbrace{\frac{1}{2}}_{\text{spin avg.}} \cdot \underbrace{\frac{1}{8}}_{\text{color avg.}} \sum_{\text{spins colors}} |\mathcal{M}|^2$$

$$= \frac{1}{2} \cdot 3 \cdot \left[0 + \frac{z^2}{(1-z)^2} + \frac{(1-z)^2}{z^2} + \frac{1}{z^2(1-z)^2} \right]$$

$$\cdot\, 2 g_s^2 q_T^2 \cdot \underbrace{2}_{(g_R \to) + (g_L \to)}$$

$$= 6\, g_s^2\, q_T^2 \quad \frac{1 + z^4 + (1-z)^4}{z^2 (1-z)^2}$$

For $q \to g\, q$, we find

$$\frac{1}{2}\frac{1}{3}\sum_{\text{spin, colors}} |m|^2 = 2 \cdot \frac{4}{3}\, g_s^2\, q_\perp^2 \quad \frac{1 + (1-z)^2}{z^2 (1-z)^2}\,(1-z)$$

This gives

$$\text{Prob}(q \to g\, q) = \frac{4}{3}\frac{\alpha_s}{\pi} \int dz \int \frac{d q_T}{q_T} \quad \frac{1 + (1-z)^2}{z}$$

j.) Changing what needs to be changed,

$$\text{Prob}(g \to g g) = 3\, \frac{\alpha_s}{\pi} \int dz \int \frac{d q_T}{q_T} \left(\frac{1 + z^4 + (1-z)^4}{z(1-z)} \right)$$