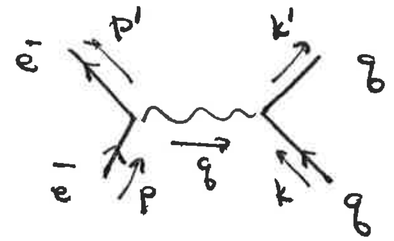
# Physics 152/252

## Problem Set #4 — Solutions

1.) a.)




$$\mathcal{M} = \langle e^-(p') | J^\mu_{EM} | e^-(p) \rangle \frac{1}{q^2} \langle q(k') | J_{EM\,\mu} | q(k) \rangle$$

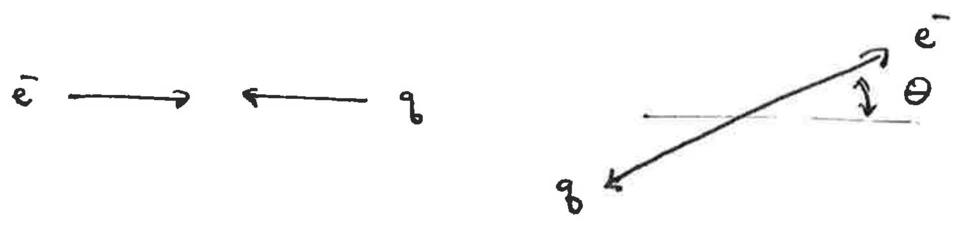
where $q^2 = t$

$$(J^\mu_{EM})_{\text{electrons}} = -e\, \bar{\psi}_e \gamma^\mu \psi_e$$

$$(J^\mu_{EM})_{\text{quarks}} = Q_f\, e\, \bar{\psi}_q \gamma^\mu \psi_q$$

$$\mathcal{M} = Q_f e^2 \langle e^- | \bar{\psi}_e \gamma^\mu \psi_e | e^- \rangle \frac{1}{t} \langle q | \bar{\psi}_q \gamma_\mu \psi_q | q \rangle$$

b.)




$p = (E, 0\ 0\ E)$ $\qquad$ $p' = (E\ \ E\sin\theta,\ 0\ \ E\cos\theta)$

$k = (E, 0\ 0 -E)$ $\qquad$ $k' = (E,\ -E\sin\theta,\ 0,\ -E\cos\theta)$

$$s = 4E^2 \qquad t = -2E^2(1-\cos\theta) \qquad u = -2E^2(1+\cos\theta)$$

c.) For $\hat{n} = (\sin\theta, 0, \cos\theta)$

$$\hat{n}\cdot\vec{S} = \tfrac{1}{2}\hat{n}\vec{\sigma} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

then

$$\tfrac{1}{2}\hat{n}\vec{\sigma}\begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} \cos\theta\cos\theta/2 + \sin\theta\sin\theta/2 \\ \sin\theta\cos\theta/2 - \cos\theta\sin\theta/2 \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix}$$

$$\tfrac{1}{2}\hat{n}\cdot\vec{\sigma}\begin{pmatrix} -\sin\theta/2 \\ \cos\theta/2 \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} -\cos\theta\sin\theta/2 + \sin\theta\cos\theta/2 \\ -\sin\theta\sin\theta/2 - \cos\theta\cos\theta/2 \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} \sin\theta/2 \\ -\cos\theta/2 \end{pmatrix}$$

so $\quad \tfrac{1}{2}\hat{n}\vec{\sigma}\,\xi_+ = +\tfrac{1}{2}\xi_+ \qquad \tfrac{1}{2}\hat{n}\cdot\vec{\sigma}\,\xi_- = -\tfrac{1}{2}\xi_-$

alternately, $\xi_+$ can be constructed by rotating around the $\hat{2}$ axis

$$\xi_+ = e^{-i\theta\sigma^2/2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix}$$

$$\xi_- = e^{-i\theta\sigma^2/2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta/2 \\ \cos\theta/2 \end{pmatrix}$$

d.) Then

| | | $\underline{R}$ | $\underline{L}$ |
|---|---|---|---|
| for $e^-$ : | | $u(p) = \sqrt{2E}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ | $\sqrt{2E}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ |
| | | $u(p') = \sqrt{2E}\begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix}$ | $\sqrt{2E}\begin{pmatrix} -\sin\theta/2 \\ \cos\theta/2 \end{pmatrix}$ |
| for $q$ : | | $u(k) = \sqrt{2E}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ | $\sqrt{2E}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ |
| | | $u(k') = \sqrt{2E}\begin{pmatrix} -\sin\theta/2 \\ \cos\theta/2 \end{pmatrix}$ | $\sqrt{2E}\begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix}$ |

e.) $\bar{\Psi}\gamma^\mu\Psi = \psi_L^\dagger \bar{\sigma}^\mu \psi_L + \psi_R^\dagger \sigma^\mu \psi_R$

so $\langle e_L^- | \bar{\Psi}\gamma^\mu\Psi | e_R^- \rangle = \langle e_R^- | \bar{\Psi}\gamma^\mu\Psi | e_L^- \rangle = 0$

for <u>massless</u> electrons

$$\langle e_R^-(p') | \bar{\Psi}\gamma^\mu\Psi | e_R^-(p) \rangle = \langle e_R^-(p') | \psi_R^\dagger \sigma^\mu \psi_R | e_R^-(p) \rangle$$

$$= u_R^\dagger(p') \left[ (1, \vec{\sigma})^\mu \right] u_R(p)$$

$$= \sqrt{2E}\, (\cos\theta/2, \sin\theta/2)\, (1, \sigma^1, \sigma^2, \sigma^3)\, \sqrt{2E} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= (\sqrt{2E})^2 (\cos\theta/2, \sin\theta/2, i\sin\theta/2, \cos\theta/2)^\mu$$

$$\langle e_L^-(p') | \bar{\Psi}\gamma^\mu\Psi | e_L^-(p) \rangle = \langle e_L^-(p') | \psi_L^\dagger \bar{\sigma}^\mu \psi_L | e_L^-(p) \rangle$$

$$= u_L^\dagger(p') \, (1, -\vec{\sigma})^\mu \, u_L(p)$$

$$= \sqrt{2E}\, (-\sin\theta/2, \cos\theta/2)\, (1, -\sigma^1, -\sigma^2, -\sigma^3)^\mu\, \sqrt{2E} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= (\sqrt{2E})^2 (\cos\theta/2, \sin\theta/2, -i\sin\theta/2, \cos\theta/2)^\mu$$

f.) $\langle q_R(k') | \bar{\Psi}\gamma^\mu\Psi | q_R(k) \rangle = \langle q_R(k') | \psi_R^\dagger \sigma^\mu \psi_R | q_R(k) \rangle$

$$= \sqrt{2E}\, (-\sin\theta/2, \cos\theta/2)\, (1, \sigma^1, \sigma^2, \sigma^3)^\mu\, \sqrt{2E} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= (\sqrt{2E})^2 (\cos\theta/2, -\sin\theta/2, i\sin\theta/2, -\cos\theta/2)^\mu$$

$$\langle q_L(k') | \bar{\Psi}\gamma^\mu\Psi | q_L(k) \rangle = \langle q_L(k') | \psi_L^\dagger \bar{\sigma}^\mu \psi_L | q_L(k) \rangle$$

$$= (\sqrt{2E})\, (\cos\theta/2, \sin\theta/2)\, (1, -\sigma^1, -\sigma^2, -\sigma^3)^\mu\, \sqrt{2E} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= (\sqrt{2E})^2 (\cos\theta/2, -\sin\theta/2, -i\sin\theta/2, -\cos\theta/2)^\mu$$

g.) Then

Lorentz product

$$\mathcal{M}(e^-_R q_R \to e^-_R q_R) = Q_f e^2 (2E)^2 \frac{1}{t}$$

$$(\cos\theta/2 \;\; \sin\theta/2 \;\; i\sin\theta/2, \; \cos\theta/2) \cdot (\cos\theta/2, -\sin\theta/2, i\sin\theta/2, -\cos\theta/2)$$

$$= \frac{Q_f e^2}{t} 4E^2 \cdot (\cos^2\theta/2 + \sin^2\theta/2 + \sin^2\theta/2 + \cos^2\theta/2)$$

$$= \frac{Q_f e^2}{t} 2s$$

$$\mathcal{M}(e^-_R q_L \to e^-_R q_L) = Q_f e^2 (2E)^2 \frac{1}{t}$$

$$(\cos\theta/2, \sin\theta/2, i\sin\theta/2, \cos\theta/2) \cdot (\cos\theta/2, -\sin\theta/2, -i\sin\theta/2, -\cos\theta/2)$$

$$= \frac{Q_f e^2}{t} \cdot 4E^2 \cdot (\cos^2\theta/2 + \sin^2\theta/2 - \sin^2\theta/2 + \cos^2\theta/2)$$

$$= \frac{Q_f e^2}{t} 8E^2 \cos^2\theta/2 = \frac{Q_f e^2}{t} 8E^2 \frac{(1+\cos\theta)}{2}$$

$$= \frac{Q_f e^2}{t} (-2u)$$

so indeed

$$|\mathcal{M}(e^-_R q_R \to e^-_R q_R)|^2 = 4Q_f^2 e^4 s^2/t^2$$

$$|\mathcal{M}(e^-_R q_L \to e^-_R q_L)|^2 = 4Q_f^2 e^4 u^2/t^2$$

2.) a.) $\langle \phi^-(p_-)\, \phi^+(p_+) |\; e j^\mu \,|0\rangle$

$$= e\,(a p_- + b p_+)^\mu\, e^{i p_-\cdot x}\, e^{i p_+\cdot x}$$

since the matrix element must be a 4-vector. Then

$$\partial_\mu \langle \phi^-\phi^+ |\, e j^\mu(x) \,|0\rangle = \langle \phi^-\phi^+ |\, e\, \partial_\mu j^\mu \,|0\rangle = 0$$

$$= e\,(p_- + p_+)\cdot(a p_- + b p_+)\, e^{i(p_- + p_+)\cdot x}$$

$$(p_- + p_+)\cdot(a p_- + b p_+) = a p_-^2 + b p_+^2 + (a+b)\, p_-\cdot p_+$$

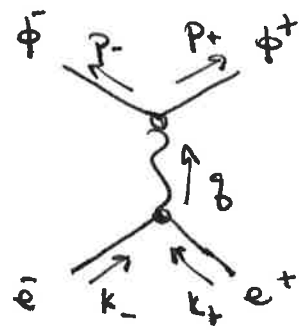
$$= (a+b)\,(m^2 + p_-\cdot p_+)$$

so $\quad a+b=0 \quad$ then

$$\langle \phi^-\phi^+ |\, e j^\mu \,|0\rangle = e\,(p_- - p_+)^\mu\, e^{+i(p_- + p_+)\cdot x} \quad \text{(const)}$$

actually, $= 1$.

b.)

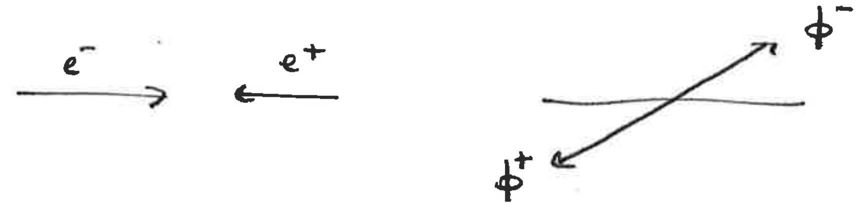



$$= e\,(p_- - p_+)^\mu\, \frac{1}{q^2}\, e\, v^\dagger(k_+)\, \sigma^\mu\, u(k_-)$$

for $e^-_R e^+_L \qquad v^\dagger \sigma^\mu u = 2E\,(0\ 1\ +i\ 0)^\mu$

for $e^-_L e^+_R \qquad v^\dagger \bar\sigma\, u = -2E\,(0\ 1\ -i,\ 0)^\mu$

c.)




$$k_- = (E, 0\ 0\ E) \qquad p_- = (E, p\sin\theta, 0\ p\cos\theta)$$

$$k_+ = (E, 0\ 0\ {-E}) \qquad p_+ = (E, -p\sin\theta, 0\ -p\cos\theta)$$

$$(p_- - p_+)^\mu = 2p\,(0\ \sin\theta, 0, \cos\theta)^\mu$$

d)

$$\mathcal{M}(e^-_R e^+_L \to \phi^-\phi^+) = \frac{e^2}{q^2}\, 2E \cdot 2p\,(-\sin\theta)$$

$$\mathcal{M}(e^-_L e^+_R \to \phi^-\phi^+) = \frac{e^2}{q^2}(-2E)(2p)(-\sin\theta)$$

$$q^2 = 4E^2$$

so

$$|\mathcal{M}(e^-_R e^+_L \to \phi^-\phi^+)|^2 = e^4 \frac{p^2}{E^2} \sin^2\theta$$

$$|\mathcal{M}(e^-_L e^+_R \to \phi^-\phi^+)|^2 = e^4 \frac{p^2}{E^2} \sin^2\theta$$

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}(e^- e^+ \to \phi^-\phi^+)|^2 = \frac{1}{2} e^4 \left(\frac{p^2}{E^2}\right) \sin^2\theta$$

$$\frac{p^2}{E^2} = 1 - \frac{m^2}{E^2} = \left(1 - \frac{4m^2}{s}\right)$$

$$\sigma = \frac{1}{2E \cdot 2E \cdot 2} \int d\Pi_2 \; \frac{1}{4} \sum |\mathcal{M}|^2$$

$$= \frac{1}{2s} \; \frac{1}{8\pi} \; \frac{P}{E} \int_{-1}^{1} \frac{d\cos\theta}{2} \; \frac{1}{2} e^4 \left(\frac{P}{E}\right)^2 \sin^2\theta$$

$$= \frac{e^4}{4 \cdot 16\pi} \left(\frac{P}{E}\right)^3 \int d\cos\theta \; \sin^2\theta$$

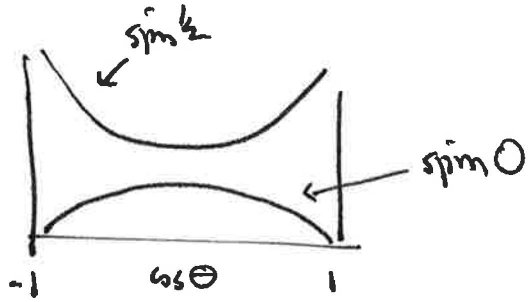
a for spin 0 particles

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{4s} \left(1 - \frac{4m_\phi^2}{s}\right)^{3/2} \cdot \sin^2\theta$$

for spin ½ particles $(m = 0)$

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{2s} \left(1 + \cos^2\theta\right)$$

so




e.) Integrate over $\theta$:

$$\int_{-1}^{1} d\cos\theta \; \sin^2\theta = \int_{-1}^{1} d\cos\theta \left(1 - \cos^2\theta\right) = 2 - \frac{2}{3} = \frac{4}{3}$$

$$\int_{-1}^{1} d\cos\theta \left(1 + \cos^2\theta\right) = 2 + 2/3 = 8/3$$

• for $m = 0$

spin 0 : $\sigma(e^+e^- \to \phi^+\phi^-) = \dfrac{\pi\alpha^2}{3s}$

spin ½ : $\sigma(e^+e^- \to f^+f^-) = \dfrac{4\pi\alpha^2}{3s}$

3.) a.)

| | $\Gamma_{tot.}$ | $BR(V \to e^+e^-)$ | $\Gamma(V \to e^+e^-)$ |
|---|---|---|---|
| $\rho^0$ | 148 | $4.72 \times 10^{-5}$ | 6.99 |
| $\omega^0$ | 8.49 | $7.28 \times 10^{-5}$ | 0.62 |
| $\phi^0$ | 4.27 | $2.95 \times 10^{-4}$ | 1.26 |
| | MeV | | keV |

b.)

$$T^i \left( \sum_a |a\rangle |\bar{a}\rangle \right)$$

$$= \sum_a \left(T^i |a\rangle\right) |\bar{a}\rangle + |a\rangle \left(T^i |\bar{a}\rangle\right)$$

$$= \sum_{ab} \left\{ t^i_{ab} |b\rangle |\bar{a}\rangle + |a\rangle \left(-(t^i)^T_{ab} |\bar{b}\rangle\right) \right\}$$

$$= \sum_{ab} \left\{ t^i_{ab} |b\rangle |\bar{a}\rangle - (t^i)_{ba} |a\rangle |\bar{b}\rangle \right\}$$

$= 0$ after renaming the indices in the second term.

c.) $|\phi^0\rangle = |s\bar{s}\rangle$

$|\rho^0\rangle$ and $|\omega^0\rangle$ are linear combinations of $|u\bar{u}\rangle$ and $|d\bar{d}\rangle$. Since $\vec{I}\,|\omega^0\rangle = 0$,

$|\omega^0\rangle$ must be: $|\omega^0\rangle = \frac{1}{\sqrt{2}}\left(|u\bar{u}\rangle + |d\bar{d}\rangle\right)$

$|\rho^0\rangle$ must be the orthogonal state: $|\rho^0\rangle = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle - |d\bar{d}\rangle)$

d.) For a quark flavor f:

$$\langle 0 | J^\mu_{EM} | f\bar{f}\rangle \propto Q_f$$

then

$$\langle 0 | J^\mu_{EM} | \phi^0\rangle = \langle 0 | J^\mu_{EM} | s\bar{s}\rangle \propto -\frac{1}{3}$$

$$\langle 0 | J^\mu_{EM} | \omega^0\rangle = \langle 0 | J^\mu_{EM} \frac{1}{\sqrt{2}}(|u\bar{u}\rangle + |d\bar{d}\rangle) \propto \frac{1}{\sqrt{2}}\left[\frac{2}{3} - \frac{1}{3}\right] = \frac{1}{\sqrt{2}}\frac{1}{3}$$

$$\langle 0 | J^\mu_{EM} | \rho^0\rangle = \langle 0 | J^\mu_{EM} \frac{1}{\sqrt{2}}(|u\bar{u}\rangle - |d\bar{d}\rangle) \propto \frac{1}{\sqrt{2}}\left[\frac{2}{3} + \frac{1}{3}\right] = \frac{1}{\sqrt{2}}$$

e.) Then $\Gamma(\rho^0) : \Gamma(\omega^0) : \Gamma(\phi^0)$

$$= |1|^2 : \left|\frac{1}{3}\right|^2 : \left|-\sqrt{2}\,\frac{1}{3}\right|^2$$

$$= 1 : \frac{1}{9} : \frac{2}{9}$$

$$= 1 : 0.11 : 0.22$$

In (a), the ratios are

$$\Gamma(\rho^0) : \Gamma(\omega^0) : \Gamma(\phi^0)$$

$$= 1 : 0.09 : 0.18$$

so the pattern is explained.