Physics 152/252

Problem Set #3 - Solutions

1.) a.) The quark model wavefunction for the proton state with $S^3 = +1/2$ is

$$|p\, S^3 = +\tfrac{1}{2}\rangle = \frac{1}{\sqrt{18}} \Big[2\,|u\!\uparrow u\!\uparrow d\!\downarrow\rangle - |u\!\uparrow u\!\downarrow d\!\uparrow\rangle - |u\!\downarrow u\!\uparrow d\!\uparrow\rangle$$
$$- |u\!\uparrow d\!\uparrow u\!\downarrow\rangle + 2\,|u\!\uparrow d\!\downarrow u\!\uparrow\rangle - |u\!\downarrow d\!\uparrow u\!\uparrow\rangle$$
$$- |d\!\uparrow u\!\uparrow u\!\downarrow\rangle - |d\!\uparrow u\!\downarrow u\!\uparrow\rangle + 2\,|d\!\downarrow u\!\uparrow u\!\uparrow\rangle \Big]$$

We need to take the matrix element in this state of

$$\vec{\mu}_B = \frac{2e\hbar}{2m_q} \left(Q_1 \vec{S}_1 + Q_2 \vec{S}_2 + Q_3 \vec{S}_3 \right)$$

If we consider the $\hat{3}$ component of this operator

$$\mu_B^3 = \frac{2e\hbar}{2m_q} \left(Q_1 \frac{\sigma_1^3}{2} + Q_2 \frac{\sigma_2^3}{2} + Q_3 \frac{\sigma_3^3}{2} \right)$$

this is diagonal in the basis above. Then, for example

$$\mu_B^3 \, |u\!\uparrow u\!\uparrow d\!\downarrow\rangle = \frac{2e\hbar}{2m_q} \left[\tfrac{2}{3}(+\tfrac{1}{2}) + \tfrac{2}{3}(+\tfrac{1}{2}) - \tfrac{1}{3}(-\tfrac{1}{2}) \right] \cdot |u\!\uparrow u\!\uparrow d\!\downarrow\rangle$$

or $\mu_B^3 |u\uparrow u\uparrow d\downarrow\rangle = \frac{e\hbar}{2m_q} \cdot \frac{5}{3} \cdot |u\uparrow u\uparrow d\downarrow\rangle$

another case is

$$\mu_B^3 |u\uparrow u\downarrow d\uparrow\rangle = \frac{2e\hbar}{2m_q}\left[\tfrac{2}{3}(+\tfrac{1}{2}) + \tfrac{2}{3}(-\tfrac{1}{2}) + (-\tfrac{1}{3})(+\tfrac{1}{2})\right] |u\uparrow u\downarrow d\uparrow\rangle$$

$$= \frac{e\hbar}{2m_q} \cdot (-\tfrac{1}{3})\, |u\uparrow u\downarrow d\uparrow\rangle$$

All other cases are related to these two by permutation symmetry.

So

$$\langle p\, S^3{=}\tfrac{1}{2} |\, \mu_B^3 \,| p\, S^3{=}\tfrac{1}{2}\rangle$$

$$= \frac{1}{18}\left[4 \cdot 3 \cdot \tfrac{5}{3} + 1 \cdot 6 \cdot (-\tfrac{1}{3})\right] \frac{e\hbar}{2m_q}$$

$$= \frac{20-2}{18} \frac{e\hbar}{2m_q} = \frac{e\hbar}{2m_q}$$

Since $\langle S^3\rangle = \frac{1}{2}$ this is

$$\left(2 \cdot \frac{m_p}{m_q}\right) \cdot \frac{e\hbar}{2m_N} \cdot \langle \vec{S}\rangle = \left(\frac{2m_p}{m_q}\right) \mu_N$$

or $\quad g_p = \frac{2m_p}{m_q}$

b.) The calculation for the neutron is simpler

$$|n\, S^3=+\tfrac{1}{2}\rangle = \frac{1}{\sqrt{18}}\Big[\, 2\,|d\uparrow d\uparrow u\downarrow\rangle - |d\uparrow d\downarrow u\uparrow\rangle - |d\downarrow d\uparrow u\uparrow\rangle$$
$$- |d\uparrow u\uparrow d\downarrow\rangle + 2\,|d\uparrow u\downarrow d\uparrow\rangle - |d\downarrow u\uparrow d\uparrow\rangle$$
$$- |u\uparrow d\uparrow d\downarrow\rangle - |u\uparrow d\downarrow d\uparrow\rangle + 2\,|u\downarrow d\uparrow d\uparrow\rangle \Big]$$

$$\mu_B^3\,|d\uparrow d\uparrow u\downarrow\rangle = \frac{2e\hbar}{2m_q}\left[-\tfrac{1}{3}(+\tfrac{1}{2}) - \tfrac{1}{3}(+\tfrac{1}{2}) + \tfrac{2}{3}(-\tfrac{1}{2})\right]|d\uparrow d\uparrow u\downarrow\rangle$$

$$= \frac{e\hbar}{2m_q}\left[-\tfrac{4}{3}\right]|d\uparrow d\uparrow u\downarrow\rangle$$

$$\mu_B^3\,|d\uparrow d\downarrow u\uparrow\rangle = \frac{2e\hbar}{2m_q}\left[-\tfrac{1}{3}(+\tfrac{1}{2}) - \tfrac{1}{3}(-\tfrac{1}{2}) + \tfrac{2}{3}(+\tfrac{1}{2})\right]|d\uparrow d\downarrow u\uparrow\rangle$$

$$= \frac{e\hbar}{2m_q}\left[\tfrac{2}{3}\right]|d\downarrow d\downarrow u\uparrow\rangle$$

then

$$\langle n\, S^3=+\tfrac{1}{2}|\,\mu_B^3\,|n\, S^3=+\tfrac{1}{2}\rangle$$

$$= \frac{1}{18}\left[4\cdot 3\cdot\left(-\tfrac{4}{3}\right) + 1\cdot 6\cdot\left(\tfrac{2}{3}\right)\right]\frac{e\hbar}{2m_q}$$

$$= \frac{-16+4}{18}\,\frac{e\hbar}{2m_q} = -\frac{2}{3}\,\frac{e\hbar}{2m_q}$$

$$= \left(-\tfrac{2}{3}\cdot 2\cdot\frac{m_p}{m_q}\right)\frac{e\hbar}{2m_p}\langle S^3\rangle$$

so

$$g_n = -\frac{4}{3}\frac{m_p}{m_q}$$

e.) We find, then, for the quark model prediction:

$$g_p = \frac{2m_p}{m_q} \qquad g_n = -\frac{4}{3}\frac{m_p}{m_q}$$

Putting in $m_q \approx \frac{1}{3}m_p \approx 300\text{ MeV}$

$$g_p \cong 6.2 \qquad g_n \cong -4.2$$

This calculation explains the large values of g_p, g_n. We could have been more sophisticated in choosing the value of m_q. Note that m_q cancels in the ratio:

$$\frac{g_p}{g_n} = -\frac{3}{2} \qquad \text{quark model}$$

$$= -1.46 \qquad \text{experiment}$$

2.) a) In the CM frame

$$Q = (E_{cm}, 0\ 0\ 0)$$

$$p_i = (E_i, \vec{p}_i)$$

$$x_1 = \frac{2E_{cm}E_1}{E_{cm}^2} = \frac{2E_1}{E_{cm}} \quad \text{similarly,} \quad x_2 = \frac{2E_2}{E_{cm}} \quad x_3 = \frac{2E_3}{E_{cm}}$$

then

$$x_1 + x_2 + x_3 = \frac{2}{E_{cm}}(E_1 + E_2 + E_3) = 2$$

b.)
$$E_i = \frac{E_{cm}}{2} x_i \qquad i = 1, 2, 3$$

$$p_i = \left[\left(\frac{E_{cm}}{2}\right)^2 x_i^2 - m_i^2\right]^{1/2}$$

so the x_i determine E_i, p_i

c.)
$$m_{12}^2 = (p_1 + p_2)^2 = (Q - p_3)^2$$

$$= Q^2 - 2Q \cdot p_3 + p_3^2$$

$$= Q^2 - Q^2 x_3 + m_3^2$$

or
$$m_{12}^2 = (1 - x_3) Q^2 + m_3^2$$

and similarly for m_{23}^2 m_{31}^3

d.) $$(p_1+p_2)^2 = m_1^2 + 2p_1\cdot p_2 + m_2^2$$
$$= m_1^2 + m_2^2 + 2E_1E_2 - p_1p_2\cos\Theta_{12} = Q^2(1-x_3) + m_3^2$$

so
$$\cos\Theta_{12} = \frac{1}{2p_1p_2}\left[2E_1E_2 + m_1^2 + m_2^2 - m_3^3 - Q^2(1-x_3)\right]$$

and, we can write every quantity on the RHS as a function of $x_1\, x_2\, x_3$

e.) $$\int d\Pi_3 = \int\frac{d^3p_1}{(2\pi)^3 2E_1}\int\frac{d^3p_2}{(2\pi)^3 2E_2}\int\frac{d^3p_3}{(2\pi)^3 2E_3}(2\pi)^4\delta^{(4)}(Q-p_1-p_2-p_3)$$

$$= \int\frac{dp_1\, p_1^2\, d\Omega_1 \quad dp_2\, p_2^2\, d\Omega_2}{(8\pi^3)^2\, 2E_1\; 2E_2\; 2E_3}(2\pi)\,\delta(E_{cm}-E_1-E_2-E_3)$$

where $\vec{p}_3 = -\vec{p}_1 - \vec{p}_2 \qquad E_3 = [|\vec{p}_3|^2 + m_3^2]^{1/2}$

Notice that E_3 depends on Θ_{12} but the other angles are free. Write

$$d\Omega_1\, d\Omega_2 = d\cos\Theta_{12}\, d\phi_{12}\; d\cos\Theta_1\, d\phi_1$$

f.) $$E_3 = \left[p_1^2 + 2p_1p_2\cos\Theta_{12} + p_2^2 + m_3^2\right]^{1/2}$$

so we can integrate

$$\int d\cos\Theta_{12}\,(2\pi)\delta(E_{cm} - E_1 - E_2 - E_3(\cos\Theta_{12}))$$

$$= 2\pi\frac{1}{\left|\frac{dE_3}{d\cos\Theta_{12}}\right|} = 2\pi\frac{1}{\frac{p_1p_2}{E_3}}$$

Now we find :

$$\int d\Pi_3 = \frac{2\pi}{64\pi^6 \cdot 8} \int \frac{dp_1\, p_1^2\, dp_2\, p_2^2}{E_1 E_2 E_3} \frac{E_3}{p_1 p_2}$$

$$\cdot \int d\phi_{12}\, d\cos\theta_1\, d\phi_1$$

g) Integrate over the remaining angles

$$\int d\phi_{12}\, d\cos\theta_1\, d\phi_1 = 2\pi \cdot 4\pi$$

h.) First $dp_1\, p_1 = dE_1\, E_1$ and similarly for $dp_2 \to dE_2$. The simplest proof is

$$d(E_1^2 - p_1^2 - m_1^2) = 0 \Rightarrow E_1 dE_1 - p_1 dp_1 = 0$$

now we have

$$\int d\Pi_3 = \frac{2\pi \cdot 2\pi \cdot 4\pi}{8 \cdot 64 \cdot \pi^6} \int dE_1\, dE_2$$

$$= \frac{1}{32\pi^3} \int dE_1\, dE_2$$

now $dE_1 = dx_1 \frac{E_{cm}}{2} = dx_1 \frac{\sqrt{Q^2}}{2}$ so

$$= \frac{Q^2}{128\pi^3} \int dx_1\, dx_2$$

also
$$d(m_{23}^2) = d\left[(1-x_1)Q^2 + m_1^2\right]$$
$$= dx_1\, Q^2$$

so

$$\int d\Pi_3 = \frac{1}{128\pi^3 Q^2} \int dm_{23}^2\, dm_{13}^2$$

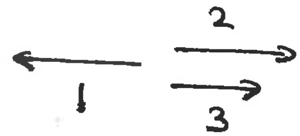
(i) $$\Gamma(A \to 1+2+3) = \frac{1}{2m_A} \int d\Pi_3\, |\mathcal{M}(A \to 123)|^2$$
$$= \frac{1}{256\pi^3 m_A^3} \int dm_{23}^2\, dm_{13}^2\, |\mathcal{M}|^2$$

If $|\mathcal{M}|^2$ is constant, the decay distribution should be <u>flat</u> in the $(m_{23}^2\ m_{13}^2)$ plane.

Note that
$$m_{23}^2 + m_{13}^2 + m_{12}^2 = Q^2(3 - x_1 - x_2 - x_3) + m_1^2 + m_2^2 + m_3^2$$
$$= Q^2 + m_1^2 + m_2^2 + m_3^2$$

so we can choose any 2 mass combinations for the integral
$$dm_{23}^2\, dm_{13}^2 = dm_{12}^2\, dm_{13}^2 = dm_{23}^2\, dm_{12}^2$$

f) For all particles massless, we can find configurations where $m_{23}^2 = 0$. There are situations where 2 and 3 are collinear



Write

$$p_1 = (E, 0\ 0\ -E)$$

$$p_2 = (zE, 0\ 0, +zE)$$

$$p_3 = ((1-z)E, 0\ 0, (1-z)E)$$

we find

$$p_1 \cdot p_2 = 2zE^2 \qquad p_1 \cdot p_3 = 2(1-z)E^2$$

$$p_2 \cdot p_3 = 0 \qquad Q^2 = (2E)^2 = 4E^2$$

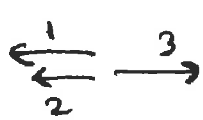
then $m_{12}^2 = 2p_1 \cdot p_2 = zQ^2 \qquad m_{13}^2 = (1-z)Q^2 \qquad 0<z<1$

$$m_{23}^2 = 0$$

Similarly for $m_{13}^2 = 0$:

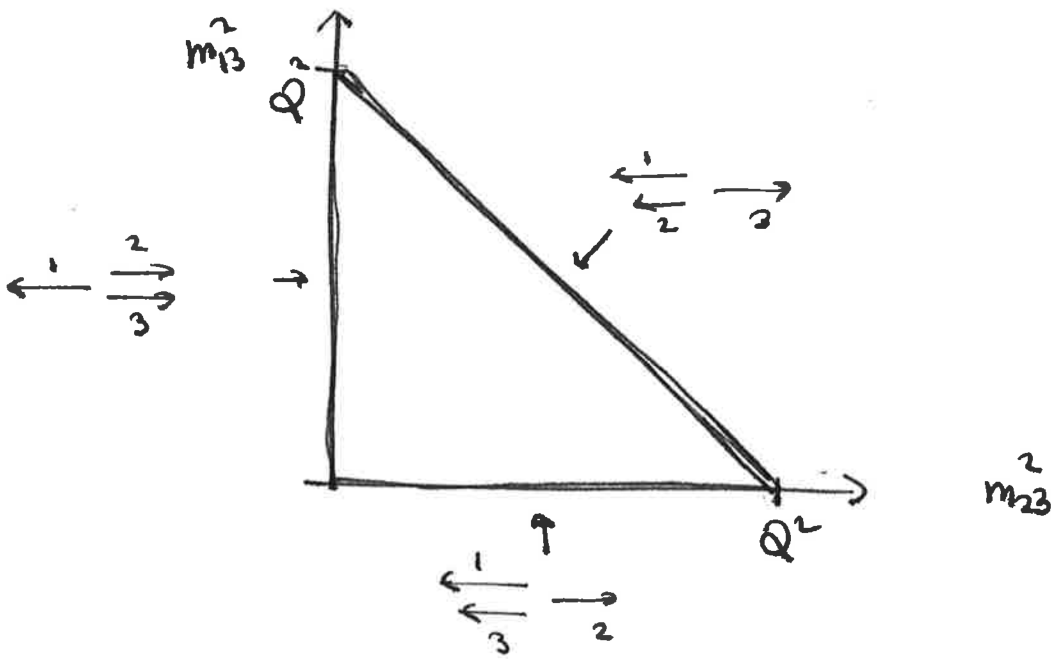
$$m_{12}^2 = zQ^2 \qquad m_{23}^2 = (1-z)Q^2$$

for $m_{12}^2 = 0$, $\quad m_{23}^2 = zQ^2 \qquad m_{13}^2 = (1-z)Q^2$



or $m_{23}^2 + m_{13}^2 = Q^2$

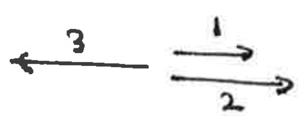
In the m_{23}^2, m_{13}^2 plane



Phase space is the <u>interior</u> of this triangle.

k) Now 1,2 are massless, 3 is massive

One boundary of the Dalitz plot is $m_{12}^2 = 0$



$$p_3 = (E_3 ,\ 0\ \ 0\ \ -p)$$

$$p_1 = (zp ,\ 0\ \ 0\ \ zp)$$

$$p_2 = ((1-z)p,\ 0\ \ 0,\ (1-z)p)$$

$$E_{cm} = E_3 + p$$

Since $(1+2)$ is a massless 4-vector, this is the situation of a massive particle recoiling against a massless particle.

Then $E_3 = \dfrac{Q^2+m_3^2}{2Q}$ $\quad p = \dfrac{Q^2-m_3^2}{2Q}$ $\quad Q = \sqrt{Q^2} = E_{cm}$

$$
\begin{aligned}
m_{13}^2 + m_{23}^2 &= (p_1+p_3)^2 + (p_2+p_3)^2 \\
&= 2p_1 \cdot p_3 + m_3^2 + 2p_2 \cdot p_3 + m_3^2 \\
&= 2(p_1+p_2)\cdot p_3 + 2m_3^2 \\
&= 2 \cdot p \cdot (E_3+p) + 2m_3^2 \\
&= 2\,\frac{Q^2-m_3^2}{2Q} \cdot Q + 2m_3^2 = Q^2 + m_3^2
\end{aligned}
$$

so one boundary is the line $\quad m_{23}^2 + m_{13}^2 = Q^2 + m_3^2$

Another boundary is given by

Write $\quad p_1 = x_1\left(\frac{Q}{2}, 0\ 0\ \frac{Q}{2}\right) \qquad p_2 = x_2\left(\frac{Q}{2}, 0\ 0\ -\frac{Q}{2}\right)$

so $\quad x_i = \dfrac{2E_i}{Q}$ as above.

$$
\begin{aligned}
p_3^2 = m_3^2 &= (Q - p_1 - p_2)^2 = Q^2 - 2Q\cdot p_1 - 2Q\cdot p_2 + (p_1+p_2)^2 \\
&= Q^2 - Q^2 x_1 - Q^2 x_2 + 2p_1\cdot p_2 \\
&= Q^2(1 - x_1 - x_2 + x_1 x_2) = Q^2(1-x_1)(1-x_2)
\end{aligned}
$$

now, from part (c)

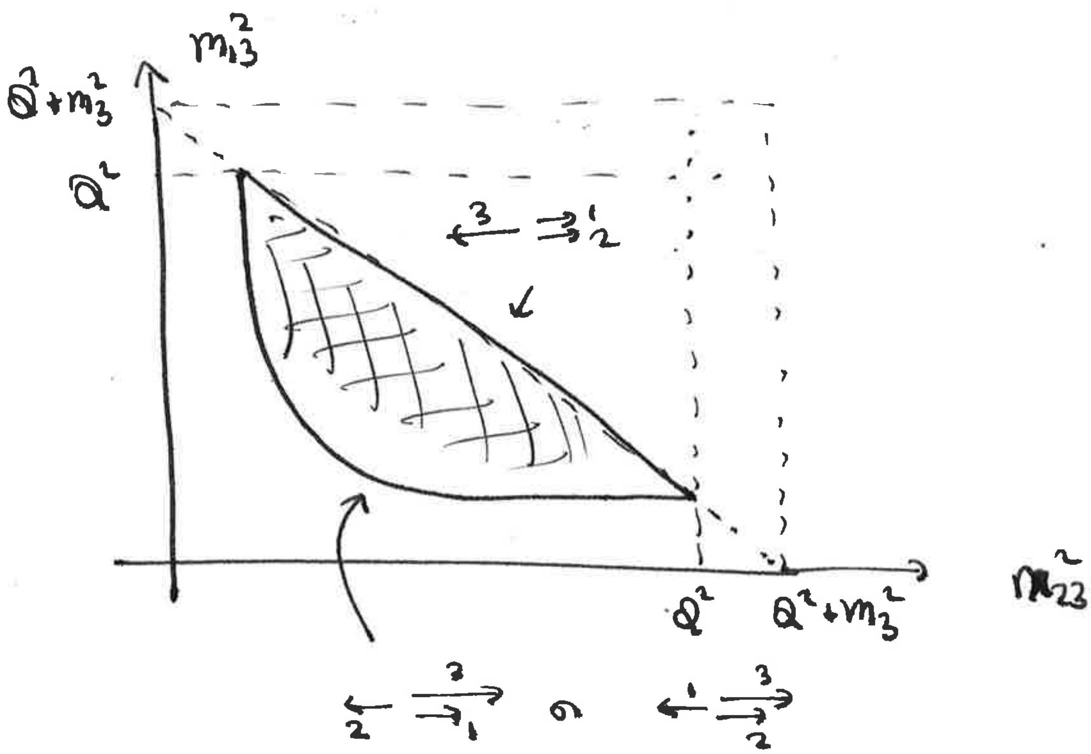
$$Q^2(1-x_1) = m_{23}^2 \qquad Q^2(1-x_2) = m_{13}^2$$

so

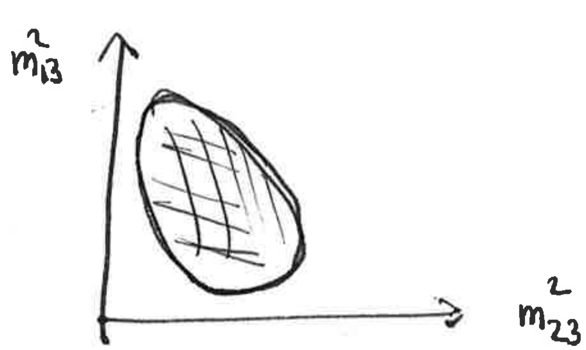
$$m_{23}^2\, m_{13}^2 = m_3^2\, Q^2$$

This is a hyperbola in the (m_{23}^2, m_{13}^2) plane.

The whole picture is



In the case in which all final particles are massive, the corners are all rounded:

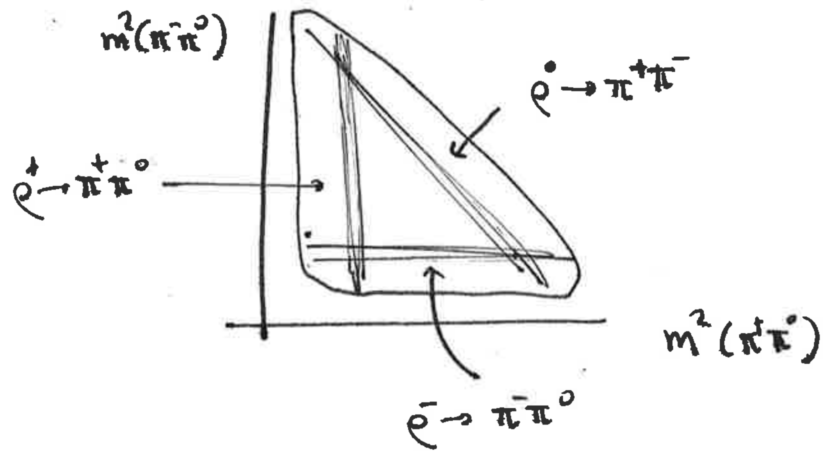


l) The three resonances are all at

$$m^2 \sim 6 \times 10^5 \text{ MeV}^2$$

and correspond to decays $A \to \pi\pi$

so $m_A \sim 770$ MeV ; this is the ρ :



k) Similarly