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Problem Set 1 - Solutions

1.) a.)

$$E_1 + E_2 = M$$

$$(m_1^2+k^2)^{1/2} + (m_2^2+k^2)^{1/2} = M$$

$$(m_1^2+k^2) + 2\left[(m_1^2+k^2)(m_2^2+k^2)\right]^{1/2} + (m_2^2+k^2) = M^2$$

$$4(m_1^2+k^2)(m_2^2+k^2) = \left[M^2 - 2k^2 - (m_1^2+m_2^2)\right]^2$$

$$\cancel{4k^4} + \cancel{4k^2(m_1^2+m_2^2)} + 4m_1^2m_2^2 = M^4 - 4M^2k^2 - 2M^2(m_1^2+m_2^2)$$
$$+ \cancel{4k^4} + \cancel{4k^2(m_1^2+m_2^2)} + (m_1^2+m_2^2)^2$$

$$4M^2k^2 = M^4 - 2M^2(m_1^2+m_2^2) + (m_1^2+m_2^2)^2 - 4m_1^2m_2^2$$

$$k^2 = \frac{1}{4M^2}\left\{M^4 - 2M^2(m_1^2+m_2^2) + (m_1^2-m_2^2)^2\right\}$$

$$k = \frac{1}{2M}\left[M^4 - 2M^2(m_1^2+m_2^2) + (m_1^2-m_2^2)^2\right]^{1/2}$$

b.) As $m_2 \to 0$ $\qquad k \to \frac{1}{2M}\left[M^4 - 2M^2 m_1^2 + m_1^4\right]^{1/2}$

$$= \frac{1}{2M}\left[(M^2 - m_1^2)^2\right]^{1/2}$$

or

$$k \to \frac{1}{2M}(M^2 - m_1^2) \qquad ✓$$

c.) $$E_1 = (k^2 + m_1^2)^{1/2}$$

$$= \frac{1}{2M}\left[M^4 - 2M^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2 + 4M^2 m_1^2\right]^{1/2}$$

$$= \frac{1}{2M}\left[M^4 + 2M^2(m_1^2 - m_2^2) + (m_1^2 - m_2^2)^2\right]^{1/2}$$

$$= \frac{1}{2M}\left[[M^2 + m_1^2 - m_2^2]^2\right]^{1/2}$$

or

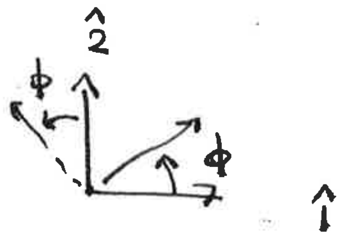
$$E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M} \qquad E_2 = \frac{M^2 + m_2^2 - m_1^2}{2M}$$

2.) a.) The angular momentum of H is a sum of

$$\underbrace{(\text{photon 1 spin})}_{\text{spin } 1} + \underbrace{(\text{photon 2 spin})}_{\text{spin } 1} + \underbrace{(\text{orbital angular momentum})}_{\text{spin } \ell}$$

All of these angular momenta are integer, so their combination is integer.

b.) Under a rotation about $\hat{3}$ by ϕ



$$\hat{1} \to \cos\phi\, \hat{1} + \sin\phi\, \hat{2}$$
$$\hat{2} \to -\sin\phi\, \hat{1} + \cos\phi\, \hat{2}$$

then

$$\vec{\epsilon}_{1R} \to \tfrac{1}{\sqrt{2}}\left((\cos\phi\, \hat{1} + \sin\phi\, \hat{2}) + i(-\sin\phi\, \hat{1} + \cos\phi\, \hat{2}) \right)$$

$$= \tfrac{1}{\sqrt{2}}\left((\cos\phi - i\sin\phi)\, \hat{1} + (\sin\phi + i\cos\phi)\, \hat{2} \right)$$

$$= e^{-i\phi}\, \tfrac{1}{\sqrt{2}}(\hat{1} + i\hat{2}) = e^{-i\phi}\, \vec{\epsilon}_{1R}$$

similarly, changing $i \leftrightarrow -i$ in the above

$$\vec{\epsilon}_{1L} = \tfrac{1}{\sqrt{2}}(\hat{1} - i\hat{2}) \longrightarrow e^{+i\phi}\, \vec{\epsilon}_{1L}$$

c.) Rotate by $180°$ about $\hat{2}$

$$\vec{\varepsilon}_{2R} = \frac{1}{\sqrt{2}}(-\hat{1} + i\hat{2}) \qquad \vec{\varepsilon}_{2L} = \frac{1}{\sqrt{2}}(-\hat{1} - i\hat{2})$$

this has $J = +1$ about $-\hat{3}$ $\qquad$ this has $J = -1$ about $-\hat{3}$

$J^3 = -1$ about $+\hat{3}$ $\qquad$ $J^3 = +1$ about $+\hat{3}$

d.) The total J^3 (about $\hat{3}$) for the 2-photon states is

RR	$\vec{\varepsilon}_{1R}\,\vec{\varepsilon}_{2R}$	$J^3 = +1 - 1 = 0$
RL	$\vec{\varepsilon}_{1R}\,\vec{\varepsilon}_{2L}$	$J^3 = +1 + 1 = +2$
LR	$\vec{\varepsilon}_{1L}\,\varepsilon_{2R}$	$J^3 = -1 - 1 = -2$
LL	$\vec{\varepsilon}_{1L}\,\vec{\varepsilon}_{2L}$	$J^3 = -1 + 1 = 0$

the cases RL, LR have $|J^3| = 2$; thus $J \geq 2$.

e.) Rotate RR by $180°$ about $\hat{2}$.

$$\frac{1}{\sqrt{2}}(\hat{1} + i\hat{2})\,\frac{1}{\sqrt{2}}(-\hat{1} + i\hat{2}) \rightarrow \frac{1}{\sqrt{2}}(-\hat{1} + i\hat{2})\frac{1}{\sqrt{2}}(+\hat{1} + i\hat{2})$$

This is a state of 2 photons with polarizations identical to the original ones. So

$$\overleftrightarrow{\varepsilon}_{RR} \rightarrow +\overleftrightarrow{\varepsilon}_{RR} \quad \text{and similarly for } \overleftrightarrow{\varepsilon}_{LL}$$

f.) The state $|J0\rangle$ transforms under angular momentum like $Y_{J0}(\theta,\varphi)$. Under a $180°$ rotation

$$Y_{J0}(\theta,\varphi) \rightarrow (-1)^J Y_{J0}(\theta,\varphi)$$

e.g. $J=1$ $\quad Y_{10}(\theta,\varphi) = C\cdot\cos\theta \rightarrow C\cdot(-\cos\theta)$

g) So, the states $\overleftrightarrow{\epsilon}_{RR}$ $\overleftrightarrow{\epsilon}_{LL}$ can have angular momentum $(Jm) = (J0)$ with $J =$ <u>even</u> only.

then, the spin of H can be $0, 2, 3 \ldots$ but not <u>1</u>.

[This argument was used to prove that the π^0 meson, which decays by $\pi^0 \rightarrow 2\gamma$, cannot be spin 1. Later, it was shown that the spin of the π^0 is 0.

More recently, this argument was used to prove that the spin of the <u>Higgs boson</u> cannot be 1.]

3.) The Dirac algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

leads to $(\gamma^0)^2 = 1 \quad (\gamma^i)^2 = -1 \qquad i = 1,2,3$

$$\gamma^0\gamma^i + \gamma^i\gamma^0 = 0 \,, \qquad \gamma^i\gamma^j + \gamma^j\gamma^i = 0 \text{ for } i \neq j$$

a.) Check some cases:

$$(\gamma^0)^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(\gamma^1)^2 = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix} = \begin{pmatrix} -(\sigma^1)^2 & 0 \\ 0 & -(\sigma^1)^2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\gamma^0\gamma^1 + \gamma^1\gamma^0 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\sigma^1 \\ -\sigma^1 & 0 \end{pmatrix} = 0$$

$$\gamma^1\gamma^2 + \gamma^2\gamma^1 = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\sigma^1\sigma^2 & 0 \\ 0 & -\sigma^1\sigma^2 \end{pmatrix} + \begin{pmatrix} -\sigma^2\sigma^1 & 0 \\ 0 & -\sigma^2\sigma^1 \end{pmatrix}$$

but $\sigma^1\sigma^2 = \sigma^3 = -\sigma^2\sigma^1$ so this is $= 0$

b.) Check some cases:

$(\gamma^0)^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (\gamma^1)^2 =$ as above $= -1$

$$\gamma^0\gamma^1 + \gamma^1\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix} + \begin{pmatrix} \sigma^1 & 0 \\ 0 & -\sigma^1 \end{pmatrix} = 0$$

$\gamma^1\gamma^2 + \gamma^2\gamma^1 =$ (as above) $= 0$

c.) $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ gives

$$U\gamma_A^0 U^\dagger = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \gamma_B^0$$

$$U\gamma_A^i U^\dagger = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \sigma^i & \sigma^i \\ -\sigma^i & \sigma^i \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 2\sigma^i \\ -2\sigma^i & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = \gamma_B^i$$