Physics 134 – Problem Set # 2

(due Tuesday, April 17)

1. This problem studies some Fourier integrals associated with Green's functions.

   (a) Evalulate the integral

   $$\int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \frac{g^2}{p^2 + a^2}$$

   Work in polar coordinates,

   $$\int d^3p = \int_0^\infty dp\, p^2 \int_{-1}^1 d\cos\theta \int_0^{2\pi} d\phi$$

   I recommend that you do the integrals over ϕ and θ first. Then you will see that the result can be rewritten as an integral over p running from $-\infty$ to ∞. View this as a contour integral in the complex plane. Since $r > 0$, you can evaluate this integral by pushing the contour upward. (Why?) Pick up the pole at $p = ia$ and show that the result is the Yukawa potential.

   (b) Evaluate the integral

   $$\int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \frac{g^2}{p^2 - k^2 - i\epsilon}$$

   Notice that, when you get to the stage of the contour integral, the contour goes *below* the pole at $p = k$ and *above* the pole at $p = -k$. Show that the result is the Green's function of the free particle Schrödinger equation with outgoing boundary conditions.

   (c) Evaluate the integral

   $$\int \frac{d^dp}{(2\pi)^d} e^{i\vec{p}\cdot\vec{x}} \frac{g^2}{p^2 + a^2}$$

   in d dimensions. This might require some poking around in the NIST Digital Library of Mathematical Functions, `http://dlmf.nist.gov`. The volume element is

   $$d^dp = dp\, p^{d-1} d\theta (\sin\theta)^{d-2} A_{d-1}\ ,$$

   The last factor is the area of the unit sphere in $(d-1)$ dimensions, which is given by (for d dimensions) by

   $$A_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}\ .$$

   (You might check the cases $d = 1, 2, 3$ to see that this makes sense. To be clear, the unit sphere in d dimensions is a $(d-1)$-dimensional surface.) The integral over θ gives a Bessel function; see the NIST Library section 10. See, in particular, section 10.11 to see how the trick of extending the p integral to $-\infty$ generalizes. Show from the explicit form of the answer that the final result falls off as e^{-ar}, like the Yukawa potential, as $r \to \infty$. What is its behavior near $r = 0$?

2. Evaluate the (first) Born approximation to the scattering amplitude for a number of sample potentials, and compare the results:

   (a) Compute the scattering amplitude in the Born approximation for the potentials:

   $$(a) \quad V(r) = Ae^{-ar}; \qquad\qquad (b) \quad V(r) = \begin{cases} B & r < a \\ 0 & r > a \end{cases}$$

   Convert these to results for $d\sigma/d\cos\theta$. Plot the results for these two potentials and for the Yukawa potential with $a = m = k = 1$ and choices of A and B such that the three functions are of comparable size. How would you distinguish these cases experimentally?

   (b) Compute the scattering amplitude in the Born approximation for a particle of charge e scattering from a charge distribution $\rho(r)$ given by

   $$(a) \quad \rho(r) = Ae^{-ar}; \qquad\qquad (b) \quad \rho(r) = \begin{cases} B & r < a \\ 0 & r > a \end{cases}$$

   with A and B chosen to give total charge e. Plot the results for $d\sigma/d\cos\theta$ for $a = 1$, $E = \frac{1}{2}$, and compare to the result for scattering from a pure Coulomb potential.

   (c) A system of zero total charge can nevertheless scatter charged particles if it has a nonzero charge density inside. Consider, for example, the charge distribution

   $$\rho(r) = e\Big[e^{-ar} - \frac{1}{8}e^{-ar/2}\Big]$$

   Plot the form factor $F(Q)$ and $d\sigma/d\cos\theta$, in the Born approximation, for scattering from this charge distribution. What property of the form factor is associated with the zero total charge of the scatterer?