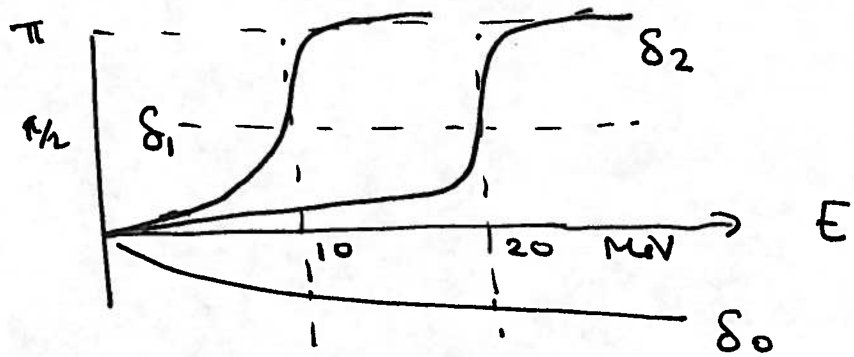
# Physics 134 - Midterm Exam Solutions

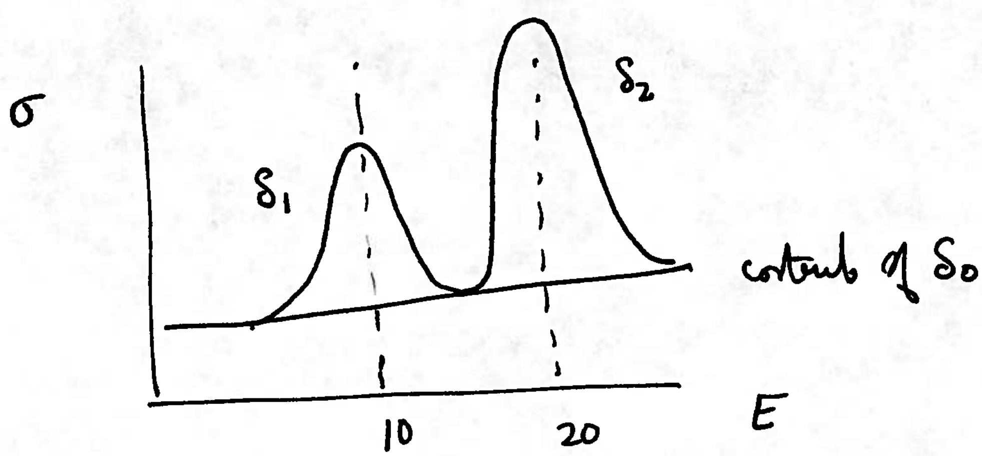
1) The behavior of the phase shifts is




a) The total cross section is

$$\sigma = \sum_\ell \frac{4\pi}{k^2}(2\ell+1)\sin^2\delta_\ell$$

$$= \frac{4\pi}{k^2}\left[(ka)^2 + 3\sin^2\delta_1 + 5\sin^2\delta_2\right]$$



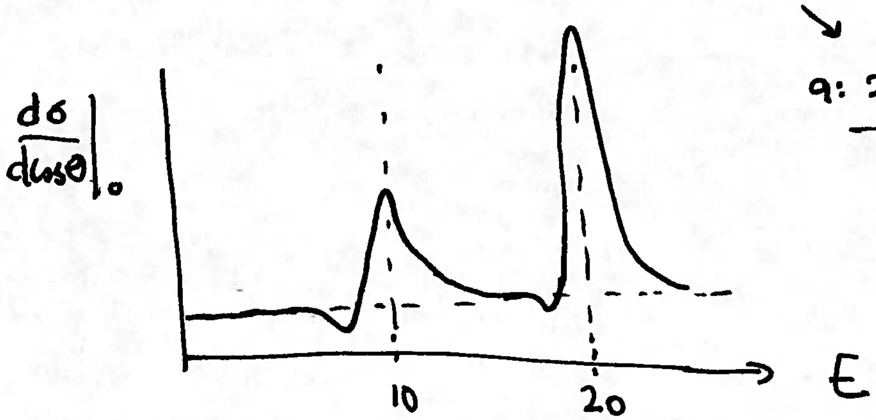
b) $$\frac{d\sigma}{d\cos\theta} = \frac{2\pi}{k^2}\left|\sum_{\ell}(2\ell+1)\, e^{i\delta_\ell}\sin\delta_\ell\, P_\ell(\cos\theta)\right|^2$$

$$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \tfrac{1}{2}(3x^2-1)$$

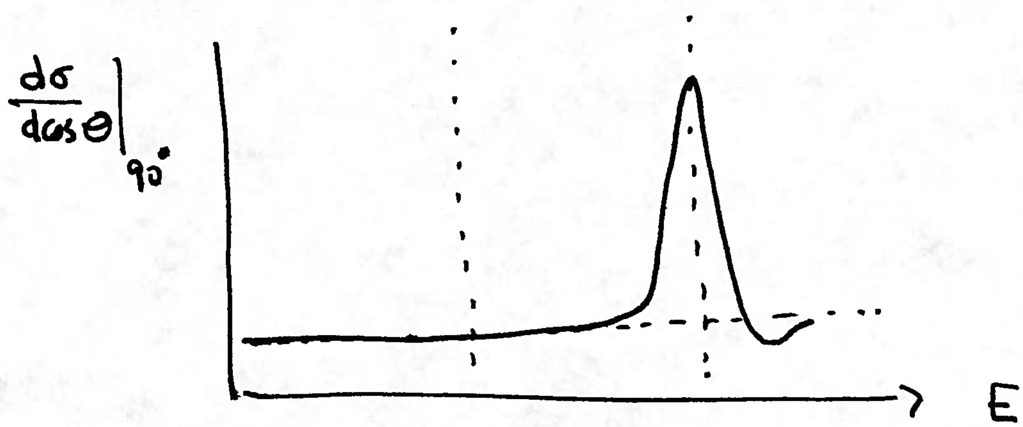
at $\theta = 0$ $\qquad P_\ell(\cos\theta) = 1$

$$\frac{d\sigma}{d\cos\theta} = \frac{2\pi}{k^2}\left| -ka + 3\, e^{i\delta_1}\sin\delta_1 + 5\, e^{i\delta_2}\sin\delta_2\right|^2$$

9: 25 ratio



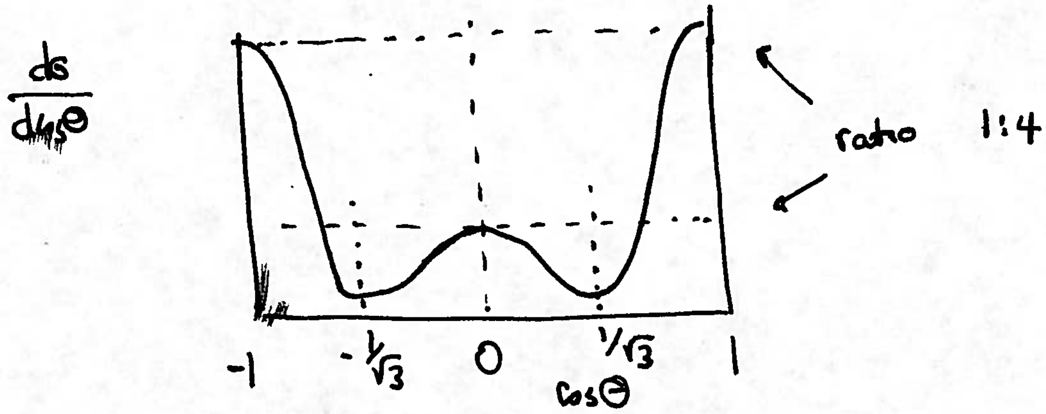

c) at $\theta = 90^\circ$ $\quad P_0 = 1 \quad P_1 = 0 \quad P_2 = -\frac{1}{2}$




Note the destructive interference below the resonances at $\theta = 0$ and above the resonance at $\theta = 90^\circ$

d) Near 20 MeV, ignore the $l=1$ component. Then

$$\frac{d\sigma}{d\cos\theta} = \frac{2\pi}{k^2}\left| -ka + 5e^{i\delta_2}\sin\delta_2\left(\frac{3\cos^2\theta-1}{2}\right)\right|^2$$




Despite my drawing, this should be symmetrical about $\cos\theta = 0$.

2.) The Schrödinger equation is

$$E\psi = -\frac{1}{2m}\frac{d^2}{dx^2}\psi + V\delta(x)\,\psi$$

integrate from $x=-\epsilon$ to $x=\epsilon$

$$\mathcal{O}(\varepsilon) = -\frac{1}{2m}\left(\frac{d\psi}{dx}\bigg|_{\varepsilon} - \frac{d\psi}{dx}\bigg|_{-\varepsilon}\right) + V\psi(0)$$

so

$$\frac{d\psi}{dx}\bigg|_{\varepsilon} - \frac{d\psi}{dx}\bigg|_{-\varepsilon} = 2mV\,\psi(0)$$

and $\psi$ itself is continuous: $\psi(\varepsilon) = \psi(-\varepsilon)$

a.) For $\varphi = \begin{cases} e^{ikx} & x<0 \\ Ae^{ikx} + Be^{-ikx} & x>0 \end{cases}$

$$A+B = 1$$

$$(ikA - ikB - ik) = 2mV$$

$$ik\,(A-B-(A+B)) = 2mV$$

$$-2ikB = 2mV$$

$$B = i\,\frac{2mV}{2k} = i\,W/2$$

$$A = 1 - i\frac{W}{2} \qquad B = i\frac{W}{2}$$

b.) Unitarity implies that the normalization of the incoming wave equals the sum of normalizations of the outgoing waves.

Then

$$|A|^2 = \underset{\text{reflected}}{|B|^2} + \underset{\text{transmitted}}{|1|^2}$$

Indeed

$$\left|1 \text{[redacted]} i\frac{W}{2}\right|^2 = \left|i\frac{W}{2}\right|^2 + 1 \quad \checkmark$$

c) For $\varphi = \begin{cases} e^{-ikx} \\ Ce^{ikx} + De^{-ikx} \end{cases}$

$$C + D = 1$$

$$ikC - ikD + ik = 2mV$$

$$+2ikC = 2mV$$

$$C = -i\frac{W}{2} \qquad D = 1 + i\frac{W}{2}$$

d)

$$T = \begin{pmatrix} 1 - i\frac{W}{2} & -i\frac{W}{2} \\ i\frac{W}{2} & 1 + i\frac{W}{2} \end{pmatrix}$$

If the wavefunction on the <u>left</u> of the delta function is

$$a e^{ikx} + b e^{-ikx}$$

then the wavefunction to the right of the delta function is

$$c e^{ikx} + d e^{-ikx}$$

where

$$\begin{pmatrix} c \\ d \end{pmatrix} = T \begin{pmatrix} a \\ b \end{pmatrix}$$

Similarly, if the wavefunction to the right of the delta function is

$$c e^{ikx} + d e^{-ikx}$$

the wavefunction to the left of the delta function is

$$\begin{pmatrix} a \\ b \end{pmatrix} = T^{-1} \begin{pmatrix} c \\ d \end{pmatrix}$$

where

$$T^{-1} = \begin{pmatrix} 1 + i W/2 & +i W/2 \\ -i W/2 & 1 - i W/2 \end{pmatrix} = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$$

For outgoing boundary conditions we set

$$d = 0 \qquad a = 1$$

Then

$$1 = a = Ec \qquad b = Gc$$

$$T = c = \frac{a}{E} \qquad R = b = \frac{G}{E} a$$

then

$$T = \frac{1}{1 + iW/2} = \frac{k}{k + imV}$$

$$R = \frac{-iW/2}{1 + iW/2} = -\frac{imV}{k + imV}$$

which is correct.

e.) Let the wavefunction to the <u>left</u> of $x = 0$ by

$$a e^{ikx} + b e^{-ikx}$$

The wavefunction immediately to the right of $x = 0$ is

$$c e^{ikx} + d e^{-ikx}$$

where

$$\begin{pmatrix} c \\ d \end{pmatrix} = T \begin{pmatrix} a \\ b \end{pmatrix}$$

If we now move to a point immediately to the left of $x = a$, and let $x' = x - a$, this becomes

$$c e^{ika} e^{ikx'} + d e^{-ika} e^{-ikx'}$$

Then the wavefunction immediately to the right of $x=a$ is

$$e\, e^{ikx'} + f\, e^{-ikx'}$$

where

$$\begin{pmatrix} e \\ f \end{pmatrix} = T \begin{pmatrix} c\, e^{ika} \\ d\, e^{-ika} \end{pmatrix}$$

$$= T \begin{pmatrix} e^{ika} & 0 \\ 0 & e^{-ika} \end{pmatrix} T \begin{pmatrix} a \\ b \end{pmatrix}$$

The inverse of this matrix is

$$\begin{pmatrix} a \\ b \end{pmatrix} = \left( T^{-1} \begin{pmatrix} e^{-ika} & \\ & e^{ika} \end{pmatrix} T^{-1} \right) \begin{pmatrix} e \\ f \end{pmatrix}$$

$$T^{-1} \begin{pmatrix} e^{-ika} & \\ & e^{ika} \end{pmatrix} T^{-1}$$

$$= \begin{pmatrix} 1+iW/2 & iW/2 \\ -iW/2 & 1-iW/2 \end{pmatrix} \begin{pmatrix} e^{-ika} & \\ & e^{ika} \end{pmatrix} \begin{pmatrix} 1+iW/2 & iW/2 \\ -iW/2 & 1-iW/2 \end{pmatrix}$$

$$= \begin{pmatrix} (1+iW/2)\, e^{-ika} & iW/2\, e^{ika} \\ -iW/2\, e^{-ika} & (1-iW/2)\, e^{ika} \end{pmatrix} \begin{pmatrix} 1+iW/2 & iW/2 \\ -iW/2 & 1-iW/2 \end{pmatrix}$$

$$= \begin{bmatrix} (1+iW/2)^2 e^{-ika} + (W/2)^2 e^{ika} & (1+iW/2)(iW/2)\, e^{-ika} + (iW/2)(1-iW/2)\, e^{ika} \\ (1+iW/2)(-iW/2)\, e^{-ika} + (-iW/2)(1-iW/2)\, e^{ika} & (W/2)^2 e^{-ika} + (1-iW/2)^2 e^{ika} \end{bmatrix}$$

using the logic of part (d)

$$T = \frac{1}{E} = \frac{1}{\left[(1+iW/2)^2 e^{-ika} + (W/2)^2 e^{ika}\right]}$$

$$R = \frac{G}{E} = \frac{(-iW/2)\left[(1+iW/2)e^{-ika} + (1-iW/2)e^{ika}\right]}{(1+iW/2)^2 e^{-ika} + (W/2)^2 e^{ika}}$$

$$(\text{transmission probability}) = \frac{1}{\left|(1+iW/2)^2 e^{-ika} + (W/2)^2 e^{ika}\right|^2}$$

$$\text{reflection probability} = \frac{W^2/4 \left|(1+iW/2)e^{-ika} + (1-iW/2)e^{ika}\right|^2}{\left|(1+iW/2)^2 e^{-ika} + (W/2)^2 e^{ika}\right|^2}$$

These expressions are somewhat complicated. But . .

f) Let $\sin\phi = \dfrac{W/2}{[1+W^2/4]^{1/2}}$ $\quad \cos\phi = \dfrac{1}{[1+W^2/4]^{1/2}}$

Then the reflection probability is proportional to

$$(1+W^2/4)\left|e^{i\phi}e^{-ika} + e^{-i\phi}e^{ika}\right|^2$$

$$= (4+W^2)\left|\cos(ka-\phi)\right|^2$$

so if

$$\phi = ka + \pi/2 + n\pi$$

The reflection probability vanishes. This is the condition

$$\tan^{-1}\frac{mV}{k} = ka + \pi/_2 + n\pi$$

(*) For the case of N delta functions, we clearly want to compute

$$T^{-1}\begin{pmatrix} e^{-ika} & \\ & e^{ika}\end{pmatrix} T^{-1}\begin{pmatrix} e^{-ika} & \\ & e^{ika}\end{pmatrix} \cdots T^{-1}$$

$$\quad 1 \quad \cdots \quad 2 \quad \cdots \quad n$$

$$= \begin{pmatrix} e^{+ika/2} & \\ & e^{-ika/2}\end{pmatrix} (V)^N \begin{pmatrix} e^{+ika/2} & \\ & e^{-ika/2}\end{pmatrix}$$

where

$$V = \begin{pmatrix} e^{-ika/2} & \\ & e^{ika/2}\end{pmatrix} T^{-1} \begin{pmatrix} e^{-ika/2} & \\ & e^{ika/2}\end{pmatrix}$$

which can be done most easily by diagonalizing $V$.