Solution Set 7

1. (a)

In hydrogen atom, the energy levels are

$$E_n = -\frac{R_y}{n^2} \Rightarrow E_0 = -R_y$$

since $R_y = \frac{1}{2}\alpha^2 m_e c^2$, $a_0 = \frac{\hbar}{m_e c \alpha}$, we have

$$R_y = \frac{1}{2}\frac{\hbar^2}{m a_0^2}$$

For hydrogen-like atom with nuclear charge Z,

$$a_0 \rightarrow \frac{a_0}{Z}$$

$$\Rightarrow E_0 \rightarrow -\frac{1}{2}\frac{\hbar^2}{m\left(\frac{a_0}{Z}\right)^2} = -Z^2 R_y$$

$\therefore$ Ionization energies of the 1S electron will be

$Z^2 R_y$

eg. $Z(Fe) = 26 \Rightarrow E_0(Fe) = -(26)^2 R_y$

$Z(Pb) = 82 \Rightarrow E_0(Pb) = -(82)^2 R_y$.

(b) In class we derived

$$\sigma(\gamma + H \to H^+ + e^-) = \frac{1024}{3}\pi\alpha\left(\frac{pa_0}{\hbar}\right)^3\left(\frac{\hbar\omega}{R_y}\right)\frac{a_0^2}{\left[1+\left(\frac{pa_0}{\hbar}\right)^2\right]^6}$$

The maximum occurs at $p = \frac{1}{\sqrt{3}}\frac{\hbar}{a_0}$.

$$\sigma_{max} = \frac{1024}{3}\pi\alpha\left(\frac{\hbar\omega}{R_y}\right)\left(\frac{1}{\sqrt{3}}\right)^3\frac{a_0^2}{\left(1+\frac{1}{3}\right)^6}$$

$$= 64\,(3)^{7/2}\,\pi\alpha\,\frac{\hbar\omega}{R_y}\,a_0^2 .$$

To find σ_{max} for the hydrogen-like atom, we make the substitutions

$$R_y \to Z^2 R_y, \quad a_0 \to Z^{-1} a_0$$

$$\Rightarrow \sigma_{max} \to 64\,(3)^{7/2}\,\pi\,\frac{\hbar\omega}{Z_1^2 R_y}\left(\frac{a_0}{Z}\right)^2$$

$$= 64\,(3)^{7/2}\,\pi\alpha\,\frac{\hbar\omega}{R_y}\,a_0^2\,Z_1^{-4}$$

$\therefore$ The dependence of σ_{max} on Z is Z^{-4}.

(c) We're approximating the atomic potential seen by the released electron as spherically symmetric square well. The photoelectric cross-section is

$$\sigma = \frac{1}{v}\int d\Pi \; |\langle e, \psi | \Delta H | 1s, Z + \gamma \rangle|^2$$

The 1s state is the appropriately rescaled 1s hydrogen wave function. The state $|e, \psi\rangle$ is the state of the freed electron in the continuum that sees the square well potential.

$$\langle e^-, \psi | \Delta H | 1s, Z + \gamma \rangle = (2\pi \alpha \omega c)^{\frac{1}{2}} \langle e^-, \psi | \vec{\varepsilon}\cdot\vec{r} | 1s, Z \rangle.$$

$\vec{\varepsilon}\cdot\vec{r}$ is $l = 1$ operator, so by angular momentum selection rule we need only look for $l = 1$ states of the electron in the continuum in the finite square well.

The radial schrodinger equation is

$$r^2 \frac{d^2}{dr^2} R_{nl} + 2r \frac{d}{dr} R_{nl} + (p^2 - 2mV_i) r^2 R_{nl} - l(l+1) R_{nl} = 0$$

$$p = \sqrt{2mE}$$

for $V(r) = \begin{cases} -\mathcal{E}, & r < R \\ 0, & r \geq R. \end{cases}$

$$R_{nl} = \begin{cases} j_l(p'r) \quad , & r<R \quad , \quad p' = \sqrt{p^2+2m\mathcal{E}} \\ A\, j_l(pr) + B\, Y_l(pr) \, , & r \geq R \end{cases}$$

Continuity at $r=R$

$$\Rightarrow \quad j_l(p'R) = A\, j_l(pR) + B\, Y_l(pR)$$

$$\Rightarrow B = \frac{j_l(p'R)}{Y_l(pR)} - A\,\frac{j_l(pR)}{Y_l(pR)}$$

$$\langle e^-, \psi | \vec{\mathcal{E}} \cdot \vec{r} \, | 1S, Z \rangle$$

$$= \int d^3r \; \overset{*}{\psi}(\vec{r}) \; \vec{\mathcal{E}} \cdot \vec{r} \; \psi_{100}^{Z}(\vec{r})$$

$$= \int r^2 dr \; R_{nl}(r)\, r\, R_{10}^{Z}(r) \;\; \vec{\mathcal{E}} \cdot \int d\Omega \; Y_{1m}^{*} \; \hat{r} \; Y_{00}$$

$$= \frac{1}{\sqrt{4\pi}} \int r^3 dr \; R_{nl}(r)\, R_{10}^{Z}(r) \;\; \vec{\mathcal{E}}^{-k} \int d\Omega \; Y_{1m}^{*} \; Y_{1k}$$

$$= \frac{1}{\sqrt{4\pi}} \int r^3 dr \; R_{nl}(r)\, R_{10}^{Z}(r) \;\; \vec{\mathcal{E}}^{-k} K_{km} \quad , \quad K_{km} = \begin{cases} -1 & \text{if } k=m=\pm 1 \\ 0 & \text{otherwise} \end{cases}$$

$$K_{km} = \begin{array}{c} \begin{array}{ccc} 1 & 0 & -1 \end{array} \\ \begin{pmatrix} -1 & & \\ & 0 & \\ & & -1 \end{pmatrix} \end{array} \begin{array}{c} \pm 1 \\ 0 \\ -1 \end{array}$$

I have defined $r^0, r^\pm, \mathcal{E}^0, \mathcal{E}^\pm$

as $\quad r^0 = (\vec{r})^3 \; , \quad r^\pm = \mp \frac{1}{\sqrt{2}}(x \pm iy)$

$\qquad \mathcal{E}^0 = (\vec{\mathcal{E}})^3 , \quad \mathcal{E}^\pm = \mp \frac{1}{\sqrt{2}} (\mathcal{E}^x \pm i \mathcal{E}^y)$

$$\Rightarrow \vec{\varepsilon} \cdot \hat{r} = \varepsilon^0 r^0 - \varepsilon^+ r^- - \varepsilon^- r^+$$

$$= \varepsilon^k r^m K_{km}$$

$$R = \int_0^\infty r^3 \, dr \; R_{n\ell}(r) \, R_{10}^{Z}(r)$$

$$R = \int_0^R r^3 \, dr \; j_1(p'r) \, \frac{2}{Z^{-3/2} a_0^{3/2}} \, e^{-Zr/a_0}$$

$$+ \int_R^\infty r^3 \, dr \; \frac{2}{Z^{-3/2} a_0^{3/2}} \, e^{-Zr/a_0} \left(A j_\ell(pr) + B Y_\ell(pr) \right)$$

We'll perform this integral on mathematica.

$$\Rightarrow |\langle e^-, \psi | \vec{\varepsilon} \cdot \vec{r} | 1s, Z \rangle|^2 = \frac{1}{4\pi} R^2 \, \vec{\varepsilon}^{\,-k} K_{km} \, \vec{\varepsilon}^{\,*-j} K_{jm}$$

$$= \frac{1}{4\pi} R^2 \, \vec{\varepsilon} \cdot \vec{\varepsilon}^{\,*} = \frac{1}{4\pi} R^2$$

$$\sigma = 2 \alpha \omega m p \int \frac{d\Omega}{4\pi} |\langle e^-, \psi | \vec{\varepsilon} \cdot \vec{r} | 1s, Z \rangle|^2$$

$$= 2 \alpha \omega m p \left(\frac{1}{4\pi} \right)^2 R^2 \int d\Omega$$

$$= \frac{\alpha \omega m p}{2\pi} R^2$$

In[108]:=
```
Clear[pp, p, Z, A, C1, C2, m, R]
int1 = FullSimplify[Integrate[r^3 SphericalBesselJ[1, pp r] Exp[-Z r / A], {r, 0, R}]]
```

Out[109]=
$$-\frac{1}{pp^2\,(A^2\,pp^2+Z^2)^3}\,A\,e^{-\frac{R\,Z}{A}}\Big(-8\,A^4\,e^{\frac{R\,Z}{A}}\,pp^3\,Z+pp\,\big(3\,A^5\,pp^4\,R+A^4\,pp^2\,(8-pp^2\,R^2)\,Z+2\,A^3\,pp^2\,R\,Z^2-2\,A^2\,pp^2\,R^2\,Z^3-A\,R\,Z^4-R^2\,Z^5\big)\,\mathrm{Cos}[pp\,R]+\big(A^5\,pp^4\,(-3+pp^2\,R^2)+5\,A^4\,pp^4\,R\,Z+2\,A^3\,pp^2\,(3+pp^2\,R^2)\,Z^2+6\,A^2\,pp^2\,R\,Z^3+A\,(1+pp^2\,R^2)\,Z^4+R\,Z^5\big)\,\mathrm{Sin}[pp\,R]\Big)$$

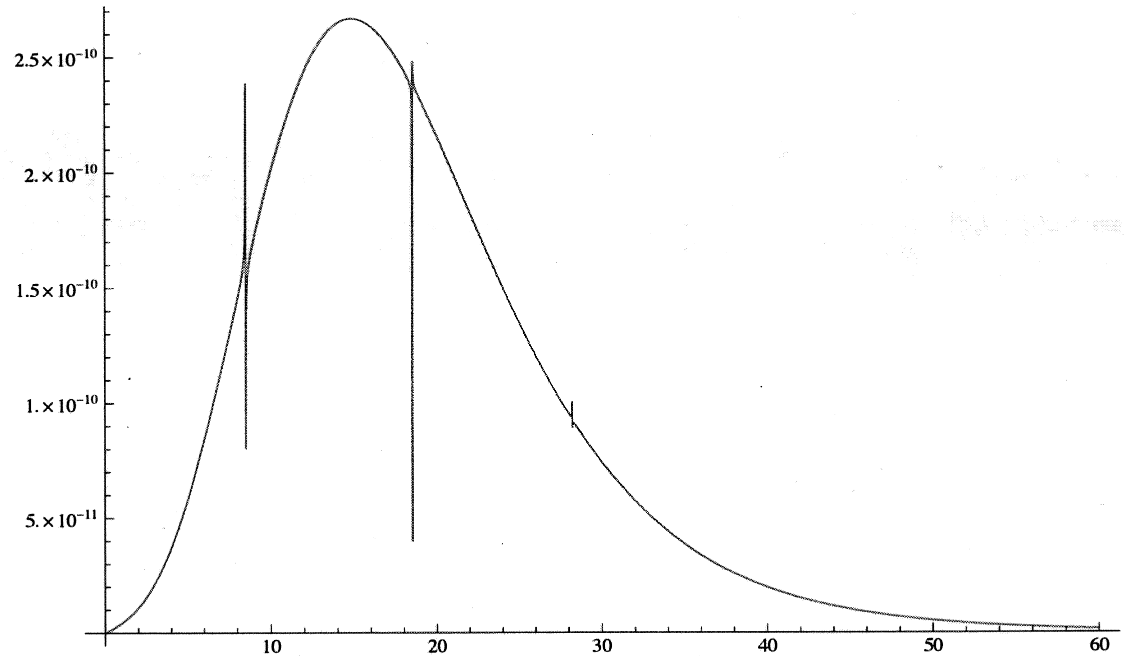
In[110]:=
```
int2 = FullSimplify[
  Integrate[r^3 * (C1 * SphericalBesselJ[1, p r] + C2 * SphericalBesselY[1, p r]) Exp[-Z r / A],
   {r, R, ∞}], Assumptions → {Im[Z] == Im[A] == Im[p] == 0, Re[A] > 0}]
```

Out[110]=
$$\mathrm{If}\Big[Z>0,\ -\frac{1}{p^2\,(A^2\,p^2+Z^2)^3}\,A\,e^{-\frac{R\,Z}{A}}\Big(\big(A^5\,p^4\,(-3\,C2-3\,C1\,p\,R+C2\,p^2\,R^2)+A^4\,p^3\,(-8\,C1+5\,C2\,p\,R+C1\,p^2\,R^2)\,Z+2\,A^3\,p^2\,(3\,C2-C1\,p\,R+C2\,p^2\,R^2)\,Z^2+2\,A^2\,p^2\,R\,(3\,C2+C1\,p\,R)\,Z^3+A\,(C2+C1\,p\,R+C2\,p^2\,R^2)\,Z^4+R\,(C2+C1\,p\,R)\,Z^5\big)\,\mathrm{Cos}[p\,R]-\big(A^5\,p^4\,(-3\,C1+3\,C2\,p\,R+C1\,p^2\,R^2)+A^4\,p^3\,(8\,C2+5\,C1\,p\,R-C2\,p^2\,R^2)\,Z+2\,A^3\,p^2\,(3\,C1+C2\,p\,R+C1\,p^2\,R^2)\,Z^2-2\,A^2\,p^2\,R\,(-3\,C1+C2\,p\,R)\,Z^3+A\,(C1-C2\,p\,R+C1\,p^2\,R^2)\,Z^4+R\,(C1-C2\,p\,R)\,Z^5\big)\,\mathrm{Sin}[p\,R]\Big),$$
$$\mathrm{Integrate}\Big[C1\,e^{-\frac{r\,Z}{A}}\,r^3\,\mathrm{SphericalBesselJ}[1,p\,r]+C2\,e^{-\frac{r\,Z}{A}}\,r^3\,\mathrm{SphericalBesselY}[1,p\,r],\ \{r,R,\infty\},\ \mathrm{Assumptions}\to \mathrm{Abs}[\mathrm{Im}[p]]\geq \mathrm{Re}\Big[\frac{Z}{A}\Big]\ ||\ \mathrm{Re}\Big[\frac{Z}{A}\Big]\leq 0\Big]\Big]$$

In[311]:=
```
A = 1;
m = 1;
EE = 5;
R = 0.33;
Z = 26;
C1 = 1;
C2 = SphericalBesselJ[1, pp R] / SphericalBesselY[1, p R] -
   C1 * SphericalBesselJ[1, p R] / SphericalBesselY[1, p R];
pp = Sqrt[p^2 + 2 * m * EE];
int = int1 + int2;
```

In[320]:= `Plot[p * int ^2, {p, 0, 60}]`

Out[320]=



2. For 1s there is no angular magnetic moment

M1 matrix element is

$$\left(\frac{\omega}{2\epsilon_0}\right)^{\frac{1}{2}} (\vec{\varepsilon}\times\hat{k})\cdot \langle J=0| \left(\frac{q_p}{2m_p c} g_p \vec{S}_p + \frac{q_e}{2m_e c} g_e \vec{S}_e\right) |J=1\rangle$$

$$= \frac{-e}{2m_e c} g_e \left(\frac{\omega}{2\epsilon_0}\right)^{\frac{1}{2}} (\vec{\varepsilon}\times\hat{k})\cdot \langle J=0| \left(\vec{S}_e - \left(\frac{g_p}{g_e}\right)\left(\frac{m_e}{m_p}\right)\vec{S}_p\right) |J=1\rangle$$

$g_e = 2$,

$$= -\frac{e}{m_e c} \left(\frac{\omega}{2\epsilon_0}\right)^{\frac{1}{2}} (\vec{\varepsilon}\times\hat{k})\cdot \langle J=0| \left(\vec{S}_e - \frac{g_p}{2}\left(\frac{m_e}{m_p}\right)\vec{S}_p\right) |J=1\rangle$$

Let $\Delta H_m^k = \langle J=0| \left(\vec{S}_e - \frac{g_p}{2}\left(\frac{m_e}{m_p}\right)\vec{S}_p\right)^k |J=1, m\rangle$,

$$= \frac{1}{\sqrt{2}}\left(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|\right)\left(\vec{S}_e - \frac{g_p}{2}\frac{m_e}{m_p}\vec{S}_p\right)^k |J=1, m\rangle$$

$$\Delta H_+^k = \frac{1}{\sqrt{2}}\left(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|\right)\left(\vec{S}_e - \frac{g_p}{2}\frac{m_e}{m_p}\vec{S}_p\right)^k |\uparrow\uparrow\rangle$$

$\Rightarrow$ $\Delta H_+^z = 0$

$$\Delta H_+^x = \frac{1}{\sqrt{2}}\left(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|\right)\left(S_e^x - \frac{g_p}{2}\frac{m_e}{m_p}S_p^x\right)|\uparrow\uparrow\rangle$$

$$S^{\pm} = S^{x} \pm i S^{y}$$

$$\Rightarrow \quad S^{x} = \frac{1}{2}\left(S^{+} + S^{-}\right) \quad , \quad S^{y} = \frac{-i}{2}\left(S^{+} - S^{-}\right)$$

$$\Rightarrow \quad \Delta H_{+}^{x} = \frac{1}{\sqrt{2}}\left(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|\right)\frac{1}{2}\left\{ \underbrace{\cancel{S_e^{+}}}_{0} + S_e^{-} - \frac{g_p}{2}\frac{m_e}{m_p}\left(\underbrace{\cancel{S_p^{+}}}_{0} + S_p^{-}\right)\right\}|\uparrow\uparrow\rangle$$

$$= \frac{1}{2\sqrt{2}}\left(\underset{e\,p}{\langle\uparrow\downarrow|} - \langle\downarrow\uparrow|\right)\left(\underset{e\,p}{|\downarrow\uparrow\rangle} - \frac{g_p}{2}\frac{m_e}{m_p}|\uparrow\downarrow\rangle\right)$$

$$= \frac{1}{2\sqrt{2}}\left(-1 - \frac{g_p}{2}\frac{m_e}{m_p}\right) = -\frac{1}{2\sqrt{2}}\left(1 + \frac{g_p}{2}\frac{m_e}{m_p}\right) \quad //$$

$$\Delta H_{+}^{y} = \frac{1}{\sqrt{2}}\left(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|\right)\left(\frac{-i}{2}\right)\left\{ -S_e^{-} + \frac{g_p}{2}\frac{m_e}{m_p}S_p^{-}\right\}|\uparrow\uparrow\rangle$$

$$= \frac{-i}{2\sqrt{2}}\left(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|\right)\left(-|\downarrow\uparrow\rangle + \frac{g_p}{2}\frac{m_e}{m_p}|\uparrow\downarrow\rangle\right)$$

$$= \frac{-i}{2\sqrt{2}}\left(1 + \frac{g_p}{2}\frac{m_e}{m_p}\right) \quad //$$

$$\Delta H_{-}^{k} = \frac{1}{\sqrt{2}}\left(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|\right)\left(\vec{S}_e - \frac{g_p}{2}\frac{m_e}{m_p}\vec{S}_p\right)^{k}|\downarrow\downarrow\rangle$$

$$\Delta H_{-}^{z} = 0$$

$$\Delta H_{-}^{x} = \frac{1}{\sqrt{2}}\left(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|\right)\frac{1}{2}\left(S_e^{+} - \frac{g_p}{2}\frac{m_e}{m_p}S_p^{+}\right)|\downarrow\downarrow\rangle$$

$$= \frac{1}{2\sqrt{2}}\left(1 + \frac{g_p}{2}\frac{m_e}{m_p}\right) \quad //$$

$$\Delta H_-^y = \frac{1}{\sqrt{2}}\left(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|\right)\left(\frac{-i}{2}\right)\left(S_e^+ - \frac{g_p}{2}\frac{m_e}{m_p}S_p^+\right)|\downarrow\downarrow\rangle$$

$$= \left(\frac{-i}{2\sqrt{2}}\right)\left(1 + \frac{g_p}{2}\frac{m_e}{m_p}\right) //$$

$$\Delta H_0^k = \frac{1}{\sqrt{2}}\left(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|\right)\left(\vec{S}_e - \frac{g_p}{2}\frac{m_e}{m_p}\vec{S}_p\right)\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

$$\Rightarrow \Delta H_0^x = \Delta H_0^y = 0$$

$$\Delta H_0^z = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\frac{g_p}{2}\frac{m_e}{m_p}\right)$$

$$= \frac{1}{4}\left(1 + \frac{g_p}{2}\frac{m_e}{m_p}\right)$$

Matrix element

$$= \frac{-e}{m_e c}\left(\frac{\omega}{2\epsilon_0}\right)^{\frac{1}{2}} \epsilon^{ijk}\varepsilon^i \hat{k}^j \Delta H_m^k$$

$$\Gamma_m = \frac{e^2}{m_e^2 c^2}\int d\Pi \sum_{\varepsilon}\left|\sqrt{\frac{\omega}{2\epsilon_0}}\ \epsilon^{ijk}\varepsilon^i\hat{k}^j\Delta H_m^k\right|^2$$

$$= \frac{e^2}{m_e^2 c^2}\left(\frac{\omega}{2\epsilon_0}\right)\frac{k^2}{\pi c}\int\frac{d\Omega}{4\pi}\sum_{\varepsilon}\left|\epsilon^{ijk}\varepsilon^i\hat{k}^j\Delta H_m^k\right|^2$$

Here ω = the frequency of the radiated photon,

$\hbar\omega$ = the splitting b/n $J=0$ & $J=1$ states.

$$\hbar\omega = \frac{8}{3} g_p \alpha^2 \frac{m_e}{m_p} Ry \qquad \text{for hydrogen}$$

$$= \frac{7}{6} \alpha^2 Ry \qquad \text{for positronium.}$$

For $\hat{k} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$,

$$\varepsilon_1 = (\cos\theta\cos\phi,\ \cos\theta\sin\phi,\ -\sin\theta),$$

$$\varepsilon_2 = (\sin\phi,\ -\cos\phi,\ 0),$$

$$\epsilon^{ijk}\varepsilon_1^i \hat{k}^j = (\sin\phi,\ -\cos\phi,\ 0)^k = \varepsilon_2^k$$

$$\epsilon^{ijk}\varepsilon_2^i \hat{k}^j = (-\cos\phi\cos\theta,\ -\sin\phi\cos\theta,\ \sin\theta)^k = -\varepsilon_1^k$$

Also, $\Delta H_+^k = \frac{1}{2\sqrt{2}}\left(1+\frac{g_p}{2}\frac{m_e}{m_p}\right)(-1,\ -i,\ 0)^k$

$$\Delta H_-^k = \frac{1}{2\sqrt{2}}\left(1+\frac{g_p}{2}\frac{m_e}{m_p}\right)(1,\ -i,\ 0)^k$$

$$\Delta H_0^k = \frac{1}{4}\left(1+\frac{g_p}{2}\frac{m_e}{m_p}\right)(0,\ 0,\ 1)^k$$

$$\Rightarrow \epsilon^{ijk}\varepsilon_1^i \hat{k}^j \Delta H_+^k = \frac{1}{2\sqrt{2}}\left(1+\frac{g_p}{2}\frac{m_e}{m_p}\right)(-\sin\phi + i\cos\phi)$$

$$= \frac{i}{2\sqrt{2}}\left(1+\frac{g_p}{2}\frac{m_e}{m_p}\right)e^{i\phi}$$

$$\epsilon^{ijk}\varepsilon_2^i \hat{k}^j \Delta H_+^k = \frac{1}{2\sqrt{2}}\cos\theta\,(\cos\phi + i\sin\phi)\left(1+\frac{g_p}{2}\frac{m_e}{m_p}\right)$$

$$= \frac{1}{2\sqrt{2}}\cos\theta\, e^{i\phi}\left(1+\frac{g_p}{2}\frac{m_e}{m_p}\right)$$

$$\sum_{\varepsilon} \left| \epsilon^{ijk} \, \varepsilon^{i} \hat{k}^{j} \Delta H_{+}^{k} \right|^{2}$$

$$= \frac{1}{8}\left(1 + \frac{g_p}{2}\frac{m_e}{m_p}\right)^2 + \frac{1}{8}\left(1 + \frac{g_p}{2}\frac{m_e}{m_p}\right)^2 \cos^2\theta$$

$$= \frac{1}{8}\left(1 + \frac{g_p}{2}\frac{m_e}{m_p}\right)^2 \left(1 + \cos^2\theta\right)$$

$$\Rightarrow \Gamma_{+} = \frac{e^2}{m_e^2 c^2}\frac{k^2}{\pi c}\left(\frac{\omega}{2\epsilon_0}\right)\int \frac{d\Omega}{4\pi}\,\frac{1}{8}\left(1 + \frac{g_p}{2}\frac{m_e}{m_p}\right)^2 \left(1 + \cos^2\theta\right)$$

$$= \frac{e^2}{m_e^2 c^5 \epsilon_0}\frac{\omega^3}{64\pi^2}\left(1 + \frac{g_p}{2}\frac{m_e}{m_p}\right)^2 \left(2\pi \, \frac{8}{3}\right)$$

$$= \left(\frac{e^2}{4\pi\epsilon_0 \hbar c}\right)\frac{\hbar\omega^3}{8 m_e^2 c^4}\left(\frac{8}{3}\right)\left(1 + \frac{g_p}{2}\frac{m_e}{m_p}\right)^2$$

Γ has units of energy (Ry) so we can insert appropriate powers of $\hbar$.

$$\Gamma_{+} = \frac{1}{3}\,\alpha\,\frac{(\hbar\omega)^3}{m_e^2 c^4}\left(1 + \frac{g_p}{2}\frac{m_e}{m_p}\right)^2 , \quad m_e^2 c^4 = \frac{4 R_y^2}{\alpha^2}$$

$$\Gamma_{+} = \frac{1}{12}\,\alpha^3\,\frac{(\hbar\omega)^3}{R_y^2}\left(1 + \frac{g_p}{2}\frac{m_e}{m_p}\right)^2$$

For Γ_-,

$$\epsilon^{ijk}\, \varepsilon_1^i\, \hat{k}^j\, \Delta H_-^k = \frac{1}{2\sqrt{2}}\left(1+\frac{g_p}{2}\frac{m_e}{m_p}\right)(\sin\phi + i\cos\phi)$$

$$\epsilon^{ijk}\, \varepsilon_2^i\, \hat{k}^j\, \Delta H_-^k = \frac{1}{2\sqrt{2}}\cos\theta\,(-\cos\phi + i\sin\phi)\left(1+\frac{g_p}{2}\frac{m_e}{m_p}\right)$$

$$\sum_{\varepsilon} \left|\epsilon^{ijk}\, \varepsilon^i\, \hat{k}^j\, \Delta H_-^k\right| = \frac{1}{8}\left(1+\frac{g_p}{2}\frac{m_e}{m_p}\right)^2 + \frac{1}{8}\cos^2\theta\left(1+\frac{g_p}{2}\frac{m_e}{m_p}\right)^2$$

$$\Rightarrow \Gamma_- = \Gamma_+ = \frac{1}{12}\alpha^3 \frac{(\hbar\omega)^3}{R_y^2}\left(1+\frac{g_p}{2}\frac{m_e}{m_p}\right)^2 \quad //$$

For Γ_0,

$$\sum_{\varepsilon} \left|\epsilon^{ijk}\, \varepsilon^i\, \hat{k}^j\, \Delta H_\theta^k\right|^2 = \frac{1}{16}\sin^2\theta\left(1+\frac{g_p}{2}\frac{m_e}{m_p}\right)^2$$

$$\int \frac{d\Omega}{4\pi} \sum_{\varepsilon} \left|\epsilon^{ijk}\, \varepsilon^i\, \hat{k}^j\, \Delta H_0^k\right|^2 = \frac{1}{24}\left(1+\frac{g_p}{2}\frac{m_e}{m_p}\right)^2$$

$$\Gamma_0 = \frac{e^2}{m_e^2 c^5 \epsilon_0}\,\frac{\omega^3}{48\pi}\left(1+\frac{g_p}{2}\frac{m_e}{m_p}\right)^2$$

$$= \frac{1}{12}\left(\frac{e^2}{4\pi\epsilon_0 \hbar c}\right)\frac{(\hbar\omega)^3\alpha^2}{4R_y^2}\left(1+\frac{g_p}{2}\frac{m_e}{m_p}\right)^2$$

$$= \frac{1}{48}\alpha^3\frac{(\hbar\omega)^3}{R_y^2}\left(1+\frac{g_p}{2}\frac{m_e}{m_p}\right)^2 \quad //$$

For hydrogen, $\hbar\omega = \frac{8}{3} g_p \alpha^2 \frac{m_e}{m_p} Ry$

$$\Rightarrow \Gamma_+ = \Gamma_- = \frac{1}{12}\alpha^3 \left(\frac{8}{3} g_p \alpha^2 \frac{m_e}{m_p}\right)^3 \left(1 + \frac{g_p}{2}\frac{m_e}{m_p}\right)^2 Ry$$

$$= \frac{128}{81}\alpha^9 g_p \left(\frac{m_e}{m_p}\right)^3 \left(1 + \frac{g_p}{2}\frac{m_e}{m_p}\right)^2 Ry$$

$$\Gamma_0 = \frac{1}{48}\alpha^3 \left(\frac{8}{3} g_p \alpha^2 \frac{m_e}{m_p}\right)^3 \left(1 + \frac{g_p}{2}\frac{m_e}{m_p}\right)^2 Ry$$

$$= \frac{32}{81}\alpha^9 g_p \left(\frac{m_e}{m_p}\right)^3 \left(1 + \frac{g_p}{2}\frac{m_e}{m_p}\right)^2 Ry$$

b. The only change in the formulas for the decay widths is that $g_p \to g_e$ and $m_p \to m_e$

$$\Rightarrow \Gamma_+ = \Gamma_- = \frac{1}{12}\alpha^3 \frac{(\hbar\omega)^3}{Ry^2}(4)$$

$$= \frac{1}{3}\alpha^3 (\hbar\omega)^3 / Ry^2$$

$$\Gamma_0 = \frac{1}{12}\alpha^3 \frac{(\hbar\omega)^3}{Ry^2}$$

$$\hbar\omega = \frac{7}{6}\alpha^2 \, Ry$$

$$\Rightarrow \Gamma_+ = \Gamma_- = \frac{1}{3}\alpha^3 \left(\frac{7}{6}\alpha^2\right)^3 Ry$$

$$= \frac{343}{648}\alpha^9 \, Ry$$

$$\Gamma_0 = \frac{343}{2592}\alpha^9 \, R_y$$

Let's go back to the formulae for the widths:

In hydrogen, all widths depend on the factor

$\left(1 + \frac{g_p}{2}\left(\frac{m_e}{m_p}\right)\right)^2$. $\frac{m_e}{m_p} \sim 10^{-3}$ so this factor is

≈ 1. For positronium on the other hand,

$\left(1 + \frac{g_e}{2}\frac{m_e}{m_{e^+}}\right)^2 = 2^2 = 4$. So we get a factor

of 4 enhancement.