# Solution Set 6

1. (a) the matrix element is

$$\Delta H_{ab} = \langle 1S | \vec{\varepsilon}^{*} \cdot \vec{r} | 3D \rangle$$

$$(\vec{\varepsilon}^{*})^a \langle n=1, l=0, m=0 | r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^a | n=3, l=2, m \rangle$$

Hydrogen atom eignstates, taking $H_0 = -\frac{\nabla^2}{2m} + \frac{e^2}{r}$, are

given by, $\psi_{nlm} = R_{nl} Y_l^m$.

$$\psi_{nlm}(r,\theta,\phi) = \sqrt{\left(\frac{2}{na_0}\right)^3 \frac{(n-l-1)!}{2n(n+l)!}}\; e^{-\rho/2} \rho^l L_{n-l-1}^{2l+1}(\rho)\, Y_l^m(\theta,\phi),$$

where $\rho = \frac{2r}{na_0}$

Going to position space,

$$\Delta H_{ab} = \int d^3r \, \langle 1S | \vec{r} \rangle \langle \vec{r} | \vec{\varepsilon}^{*} \cdot \vec{\underset{\sim}{r}} | 3D, m \rangle$$

$$= (\vec{\varepsilon}^{*})^a \int d^3r \, \langle 1S | \vec{r} \rangle \, \vec{r} \, \langle \vec{r} | 3D, m \rangle$$

$$= (\vec{\varepsilon}^{*})^a \int d^3r \, \psi_{100}^{*} \, \psi_{32m} (\vec{r})^a$$

$$= (\vec{\varepsilon}^{*})^a \int r^2 dr \, d\Omega \, \psi_{100}^{*} \, \psi_{32m} \, r(\hat{r})^a$$

$$= (\vec{\varepsilon}^{*})^a \int dr \, r^3 \, R_{10}^{*} R_{32} \int d\Omega \, Y_{00}^{*} \, Y_{2m} (\hat{r})^a$$

$$\int d\Omega \, Y_{00}^* \, Y_{2m} \, (\hat{r})^a = \sqrt{\frac{1}{4\pi}} \int d\Omega \, Y_{2m} \, (\hat{r})^a$$

Noting that $\hat{r} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$$= \sqrt{\frac{8\pi}{3}} \left( -Y_{11} + Y_{1-1} \, , -Y_{11} - Y_{1-1} \, , \frac{1}{\sqrt{2}} Y_{10} \right),$$

and the orthogonality of the $Y_{\ell m}$'s, $\int d\Omega \, Y_{\ell m} Y_{\ell' m'} = \delta_{\ell\ell'}\delta_{mm'}$,

we see that $\int d\Omega \, Y_{2m}(\hat{r})^a = 0$

For decay to 2S states, $\psi_{200} = R_{20} Y_{00}$ and we get the integral over solid angle to be the same! Therefore the electric dipole has no matrix element between any of the 3D states and 1S or 2S states.

Decays to 2P states however will have matrix element of the dipole operator $\sim \int d\Omega \, Y_{1m}^* \, Y_{2m'} (\hat{r})^a$ and these are non zero!

(b) We want $\langle 2P, m | \vec{\varepsilon}^* \cdot \vec{r} | 3D, m' \rangle$ for various m and m's.

Let $\Delta H_{mm'} = \langle 2P, m | \vec{\varepsilon}^* \cdot \vec{r} | 3D, m' \rangle$,

$$\Delta H_{mm'} = (\vec{\varepsilon}^*)^a \int dr\, r^3 R_{21}^* R_{32} \int d\Omega\, Y_{1m}^* Y_{2m'} (\hat{r})^a \qquad (1)$$

Let's perform the angular integration first.

Let $r^\pm = \mp \left\{ \frac{\hat{r}_x \pm i \hat{r}_y}{\sqrt{2}} \right\} \Rightarrow$

$$= \mp \frac{1}{\sqrt{2}} (\sin\theta\cos\phi \pm i \sin\theta \sin\phi)$$

$$= \mp \frac{1}{\sqrt{2}} \sin\theta\, e^{\pm i\phi}$$

$$= \frac{\pm}{\sqrt{2}} \sqrt{\frac{8\pi}{3}}\, Y_{1,\pm 1}$$

$$= \sqrt{\frac{4\pi}{3}}\, Y_{1,\pm 1} ,$$

Also let $r^0 = \hat{r}_z = \cos\theta = \sqrt{\frac{4\pi}{3}}\, Y_{10}$

Let $\tilde{C}^q_{mm'} = \int d\Omega\, Y_{1m}^* Y_{2m'} \sqrt{\frac{4\pi}{3}}\, Y_{1q}$

$$= \sqrt{\frac{4\pi}{3}} \int d\Omega\, Y_{1m}^* Y_{2m'} Y_{1q}$$

Performing the $\phi$ integral first, we immediately see that $\tilde{C}^{q}_{mm'} = 0$ unless $m = m' + q$. The non-zero elements are:

$$\tilde{C}^{0}_{00}, \quad \tilde{C}^{1}_{0,-1} \quad \tilde{C}^{-1}_{0\,1};$$

$$\tilde{C}^{0}_{1\,1}, \quad \tilde{C}^{1}_{1\,0}, \quad \tilde{C}^{-1}_{1\,2}$$

$$\tilde{C}^{0}_{-1\,-1}, \quad \tilde{C}^{-1}_{-1\,0}, \quad \tilde{C}^{1}_{-1\,-2}$$

We can proceed now to calculate their values:

$$\tilde{C}^{0}_{00} = \sqrt{\frac{4\pi}{3}} \int d\Omega\, Y^{*}_{10}\, Y_{20}\, Y_{10}$$

$$= \sqrt{\frac{4\pi}{3}}\, (2\pi) \left(\frac{3}{4\pi}\right) \int d(\cos\theta)\, \cos^2\theta \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$= \sqrt{4\pi^2} \sqrt{\frac{3}{4\pi}} \sqrt{\frac{5}{4\pi}} \int d(\cos\theta) \left(\frac{3}{2}\cos^4\theta - \cos^2\theta\right)$$

$$= \sqrt{\frac{15}{4}}\, \frac{4}{15}$$

$$= \sqrt{\frac{4}{15}}$$

We may check our answer the following way:

$$\langle 2P, m |\, \vec{\varepsilon}^{*} \cdot \vec{r}\, | 3D, m' \rangle$$

$$= \vec{\varepsilon}^{*} \cdot \langle 2, 1, m |\, \vec{r}\, | 3, 2, m' \rangle$$

$=$

Note that $\vec{r}$ is a rank 1 spherical tensor:

$$T_1^0 = z \,, \quad T_1^{\pm} = \mp \frac{1}{\sqrt{2}} (x \pm iy)$$

Also, defining $\varepsilon^{*\pm} = \mp\frac{1}{\sqrt{2}} ( \varepsilon^*_x \pm i \varepsilon^*_y ) \,, \; \varepsilon^{*0} = \varepsilon^*_z \,,$

We can check that $\vec{\varepsilon}^* \cdot \vec{r} = \varepsilon^{*0} T_1^0 + \varepsilon^{*1} T_1^{-1} + \varepsilon^{*-1} T_1^1$

$$= \varepsilon^{*-k} T_1^{\,k} \quad //$$

Then $\vec{\varepsilon}^* \cdot \langle 2,1,m | \vec{r} \,| 3,2,m' \rangle$

$$= \varepsilon^{*-k} \langle 2,1,m | T_1^k | 3,2,m' \rangle$$

Clebsch-Gordon coefficient.

$$= \varepsilon^{*-k} \langle 2,1 | T_1 | 3\,2 \rangle \langle 1,m | 1,k; 2\, m' \rangle$$

using Wigner-Eckart.

$$= \langle 2,1 \| T_1 \| 3\,2 \rangle \left\{ \varepsilon^{*-k} \langle 1,m | 1,k; 2\, m' \rangle \right\} \quad // \qquad (2)$$

We claim now that up to an overall factor B,

$$\langle 2\,1 \| T_1 \| 3\,2 \rangle = B \int r^3 dr\, R_{21}^* R_{32} \,,$$

$$\varepsilon^{*-k} \langle 1,m | 1,k; 2,m' \rangle = \frac{1}{B} \varepsilon^{*-k} \int d\Omega\, Y_{1m}^* Y_{2m'} (\hat{r})^k$$

$$= \frac{1}{B} \varepsilon^{*-k} \tilde{C}^k_{mm'}$$

$$\Rightarrow \langle 1,m | 1,k; 2,m' \rangle = \frac{1}{B} \tilde{C}^k_{mm'}$$

so that (1) = (2).

From the above calculation, we see that

$$\tilde{C}^{0}_{00} = B \langle 1, 0 | 1, 0; 2, 0 \rangle$$

$$\Rightarrow \sqrt{\frac{4}{15}} = B\left(-\sqrt{\frac{2}{5}}\right)$$

$$\Rightarrow B = -\sqrt{\frac{2}{3}}.$$

So we expect

$\tilde{C}^{k}_{mm'} = -\sqrt{\frac{2}{3}} \langle 1\, m | 1\, k; 2\, m' \rangle$. Let's proceed to show this.

$$\tilde{C}^{1}_{0-1} = \sqrt{\frac{4\pi}{3}} \int d\Omega\, Y^{*}_{10} Y_{2-1} Y_{11}$$

$$= \sqrt{\frac{4\pi}{3}} \int d\Omega \sqrt{\frac{3}{4\pi}} \cos\theta \sqrt{\frac{15}{8\pi}} \sin\theta\cos\theta\, e^{-i\phi} \left(-\sqrt{\frac{3}{8\pi}}\right) \sin\theta\, e^{i\phi}$$

$$= -\sqrt{\frac{15}{8\pi}} \sqrt{\frac{3}{8\pi}} \sqrt{4\pi^2} \int d(\cos\theta) \cos^2\theta \sin^2\theta$$

$$= -\sqrt{\frac{45}{16}}\ \frac{4}{15}$$

$$= -\sqrt{\frac{1}{5}}$$

On the other hand, $-\sqrt{\frac{2}{3}} \langle 1\, 0 | 1\, 1; 2\, -1 \rangle = -\sqrt{\frac{2}{3}} \sqrt{\frac{3}{10}} = -\sqrt{\frac{1}{5}}$ ✓

Let's do one more integrals, for $m = 1$ and compute the rest from Clebsch-Gordon coefficients.

$$\tilde{C}^{1}_{10} = \sqrt{\frac{4\pi}{3}} \int d\Omega \, Y^{*}_{11} Y_{20} Y_{11}$$

$$= \sqrt{\frac{4\pi}{3}} \int d\Omega \left(-\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{-i\phi}\right) \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) \left(-\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{i\phi}\right)$$

$$= \sqrt{\frac{4\pi}{3}} \sqrt{\frac{3}{8\pi}} \sqrt{\frac{3}{8\pi}} \sqrt{\frac{5}{4\pi}} \, 2\pi \int d(\cos\theta) \sin^2\theta \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$= \sqrt{\frac{15}{16}} \left(-\frac{4}{15}\right)$$

$$= -\sqrt{\frac{1}{15}}$$

$$-\sqrt{\frac{2}{3}} \langle 1 1 | 1 1 ; 2 0 \rangle = -\sqrt{\frac{2}{3}} \sqrt{\frac{1}{10}} = -\sqrt{\frac{1}{15}} \quad \checkmark$$

$$\tilde{C}^{-1}_{01} = -\sqrt{\frac{2}{3}} \langle 1 0 | 1 -1 ; 2, 1 \rangle$$

$$= -\sqrt{\frac{2}{3}} \sqrt{\frac{3}{10}} = -\sqrt{\frac{1}{5}} \ //$$

$$\tilde{C}^{0}_{11} = -\sqrt{\frac{2}{3}} \langle 1 1 | 1 0 ; 2 1 \rangle$$

$$= -\sqrt{\frac{2}{3}} \left(-\sqrt{\frac{3}{10}}\right) = \sqrt{\frac{1}{5}} \ //$$

$$\tilde{C}^{0}_{-1-1} = -\sqrt{\frac{2}{3}} \langle 1 -1 | 1 0 ; 2 -1 \rangle$$

$$= -\sqrt{\frac{2}{3}} \left(-\sqrt{\frac{3}{10}}\right) = \sqrt{\frac{1}{5}} \ //$$

$$\tilde{C}^{-1}_{-10} = -\sqrt{\frac{2}{3}} \langle 1 -1 | 1 -1 ; 2 0 \rangle = -\sqrt{\frac{2}{3}} \sqrt{\frac{1}{10}} = -\sqrt{\frac{1}{15}} \ //$$

$$\tilde{C}^{1}_{-1-2} = -\sqrt{\frac{2}{3}} \langle 1 -1 | 1 1 ; 2 -2 \rangle = -\sqrt{\frac{2}{3}} \sqrt{\frac{3}{5}} = -\sqrt{\frac{2}{5}} \ //$$

$$\tilde{C}^{-1}_{12} = -\sqrt{\frac{2}{3}} \langle 1 1 | 1 -1 ; 2, 2 \rangle = -\sqrt{\frac{2}{3}} \sqrt{\frac{3}{5}} = -\sqrt{\frac{2}{5}} \ //$$

To calculate the matrix element $\Delta H_{mm'}$ we need to do the radial integration

$$\int dr\, r^3 R_{21}^* R_{32}$$

$$= \int dr\, r^3 \frac{1}{\sqrt{24}\, a_0^{3/2}} \frac{r}{a_0} e^{-r/2a_0} \frac{4}{81\sqrt{30}\, a_0^{3/2}} \left(\frac{r}{a_0}\right)^2 e^{-r/3a_0}$$

$$= \frac{4}{81\sqrt{24}\sqrt{30}} \frac{1}{a_0^6} \int dr\, r^6 e^{-r/2a_0 - r/3a_0}$$

$$= \frac{1}{3^5\sqrt{5}} \frac{1}{a_0^6} \int dr\, r^6 e^{-r\left(\frac{5}{6a_0}\right)}$$

$$= \frac{1}{3^5\sqrt{5}} \frac{1}{a_0^6} \left(\frac{6}{5}a_0\right)^7 \int dr'\, r'^6 e^{-r'} \quad , \quad r' = \frac{5}{6a_0} r$$

$$= \underline{\underline{3^2 2^7 5^{-15/2}\, 7!\, a_0}}$$

$$\Rightarrow \Delta H_{mm'} = 3^2 2^7 5^{-15/2}\, 7!\, a_0\, \vec{\varepsilon}^* \cdot \vec{\tilde{C}}_{mm'}$$

Defining $\varepsilon^{*k}$, $k = 0, +1, -1$ as $\varepsilon^{*0} = \varepsilon_z^*$, $\varepsilon^{*\pm} = \mp\frac{1}{\sqrt{2}}(\varepsilon_x^* \pm i\varepsilon_y^*)$

We see that $\vec{\varepsilon}^* \cdot \vec{\tilde{C}}_{mm'} = \varepsilon^{*0}\tilde{C}^0_{mm'} - \varepsilon^{*+}\tilde{C}^-_{mm'} - \varepsilon^{*-}\tilde{C}^+_{mm'}$.

Note the minus signs! $= \varepsilon^{*0}\tilde{C}^0_{mm'} + \varepsilon^-\tilde{C}^-_{mm'} + \varepsilon^+\tilde{C}^+_{mm'}$

Let $C_{mm'} = \varepsilon^{*-k}\tilde{C}^k_{mm'} = \varepsilon^{*0}\tilde{C}^0_{mm'} - \varepsilon^{*+}\tilde{C}^-_{mm'} - \varepsilon^{*-}\tilde{C}^+_{mm'}$.

so that $\Delta H_{mm'} = \frac{3^2 2^7\, 7!}{5^{15/2}} a_0 C_{mm'}$, we find the following values for $C$:

$$\boxed{\begin{aligned} C_{00} &= \varepsilon^{*0}\tilde{C}^0_{00} = \sqrt{\tfrac{4}{15}}\,\varepsilon^{*0}, & C_{01} &= -\varepsilon^{*+1}\tilde{C}^{-1}_{01} = +\sqrt{\tfrac{1}{5}}\,\varepsilon^{*+1} \\ C_{10} &= -\varepsilon^{*-}\tilde{C}^{+1}_{10} = +\sqrt{\tfrac{1}{15}}\,\varepsilon^{*-}, & C_{11} &= \varepsilon^{*0}\tilde{C}^0_{11} = \sqrt{\tfrac{1}{5}}\,\varepsilon^{*0} \end{aligned}}$$

$$C_{0-1} = -\varepsilon^{*-}\,\tilde{C}^{1}_{0-1} = +\sqrt{\tfrac{1}{5}}\,\varepsilon^{*-}, \quad C_{-10} = -\varepsilon^{*+}\,\tilde{C}^{-1}_{-10} = +\sqrt{\tfrac{1}{15}}\,\varepsilon^{*+}$$

$$C_{-1-1} = \varepsilon^{*0}\,\tilde{C}^{0}_{-1-1} = \sqrt{\tfrac{1}{5}}\,\varepsilon^{*0}, \quad C_{-1-2} = -\varepsilon^{*-1}\,\tilde{C}^{+1}_{-1-2} = +\sqrt{\tfrac{2}{5}}\,\varepsilon^{*-}$$

$$C_{12} = -\varepsilon^{*+}\,\tilde{C}^{-1}_{12} = +\sqrt{\tfrac{2}{5}}\,\varepsilon^{*+}$$

$$\Gamma_{m'} = \int d\Pi \sum_m \left| \langle 2P, m + \gamma \,|\, \Delta H \,|\, 3D, m' \rangle \right|^2$$

Note: Since the final states are distinct we shoud sum afte squaring.

$$= \frac{2\alpha\omega^3}{c^2} \int \frac{d\Omega}{4\pi} \sum_m \left| \langle 2P, m \,|\, \vec{\varepsilon}^{\,*}\cdot\vec{r} \,|\, 3D, m' \rangle \right|^2$$

$$= \frac{2\alpha\omega^3}{c^2} \int \frac{d\Omega}{4\pi} \sum_m \left| \Delta H_{mm'} \right|^2$$

$$= \frac{2\alpha\omega^3}{c^2} \, \frac{3^4\, 2^{14}\, (7!)^2}{5^{15}}\, a_0^2 \int \frac{d\Omega}{4\pi} \sum_m \left| C_{mm'} \right|^2$$

$$= \Gamma\, \gamma_{m'}, \text{ where } \boxed{\gamma_{m'} \doteq \int \frac{d\Omega}{4\pi} \sum_m \left| C_{mm'} \right|^2}, \quad \boxed{\Gamma = \frac{2\alpha\omega^3}{c^2}\, \frac{3^4\, 2^{14}\, (7!)^2}{5^{15}}\, a_0^2}$$

$$\gamma_2 = \int \frac{d\Omega}{4\pi} \left| \sqrt{\tfrac{2}{5}}\, \varepsilon^{*+} \right|^2$$

The solid-angle integral is over the photon's solid angle.

let $\hat{k} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$, then the two orthogonal polarization vectors to this $\hat{k}$ are

$$\hat{\varepsilon}_1 = (\cos\theta\cos\phi,\ \cos\theta\sin\phi,\ -\sin\theta),$$

$$\hat{\varepsilon}_2 = (\sin\phi, -\cos\phi, 0).$$

$$\Rightarrow \quad \varepsilon_1^0 = -\sin\theta \qquad\qquad \Rightarrow \quad \boxed{\begin{aligned} \varepsilon_1^{*0} &= -\sin\theta \\ \varepsilon_1^{*+} &= -\tfrac{1}{\sqrt{2}}\cos\theta\, e^{-i\phi} \\ \varepsilon_1^{*-} &= \tfrac{1}{\sqrt{2}}\cos\theta\, e^{i\phi} \end{aligned}}$$

$$\varepsilon_1^+ = -\frac{1}{\sqrt{2}}(\varepsilon_1^x + i\varepsilon_1^y) = -\frac{1}{\sqrt{2}}\cos\theta\, e^{i\phi}$$

$$\varepsilon_1^- = \frac{1}{\sqrt{2}}(\varepsilon_1^x - i\varepsilon_1^y) = \frac{1}{\sqrt{2}}\cos\theta\, e^{-i\phi}$$

Also,

$$\varepsilon_2^0 = 0$$

$$\varepsilon_2^+ = -\frac{1}{\sqrt{2}}(\sin\phi - i\cos\phi) = \frac{i}{\sqrt{2}}e^{i\phi}$$

$$\varepsilon_2^- = \frac{1}{\sqrt{2}}(\sin\phi + i\cos\phi) = \frac{i}{\sqrt{2}}e^{-i\phi}$$

$$\boxed{\begin{aligned} \varepsilon_2^{*0} &= 0 \\ \varepsilon_2^{*+} &= -\tfrac{i}{\sqrt{2}}e^{-i\phi} \\ \varepsilon_2^{*-} &= -\tfrac{i}{\sqrt{2}}e^{i\phi} \end{aligned}}$$

We'll sum over the polarizations of the final state photon. However, this time we do the sum after we square the amplitudes.

$$\Rightarrow \quad \gamma_2 = \sum_{\varepsilon} \int \frac{d\Omega}{4\pi} \left| +\sqrt{\frac{2}{5}}\, \varepsilon^{*+} \right|^2$$

$$= \int \frac{d\Omega}{4\pi} \left(\frac{2}{5}\right) \left\{ |\varepsilon_1^{*+}|^2 + |\varepsilon_2^{*+}|^2 \right\}$$

$$= \frac{1}{10\pi} \int d\Omega \left\{ \frac{1}{2}\cos^2\theta + \frac{1}{2} \right\}$$

$$= \frac{1}{10} \int d\cos\theta\, (\cos^2\theta + 1) = \frac{4}{15} \quad //$$

$$\gamma_{-2} = \sum_{\varepsilon} \int \frac{d\Omega}{4\pi} \sum_m \left| C_{m\,-2} \right|^2$$

$$= \sum_{\varepsilon} \int \frac{d\Omega}{4\pi} \left| +\sqrt{\tfrac{2}{5}}\ \varepsilon^{*-} \right|^2$$

$$= \int \frac{d\Omega}{4\pi} \left(\frac{2}{5}\right) \left\{ |\varepsilon_1^-|^2 + |\varepsilon_2^-|^2 \right\}$$

$$= \frac{1}{10} \int d\cos\theta \left\{ \tfrac{1}{2}\cos^2\theta + \tfrac{1}{2} \right\}$$

$$= \frac{4}{15} //$$

$$\gamma_1 = \sum_{\varepsilon} \int \frac{d\Omega}{4\pi} \left\{ |C_{01}|^2 + |C_{11}|^2 \right\}$$

$$= \sum_{\varepsilon} \int \frac{d\Omega}{4\pi} \left| +\sqrt{\tfrac{1}{5}}\ \varepsilon^{*+} + \sqrt{\tfrac{1}{5}}\ \varepsilon^{*0} \right|^2$$

$$= \sum_{\varepsilon} \int \frac{d\Omega}{4\pi} \frac{1}{5} \left\{ |+\varepsilon^{*+}|^2 + |\varepsilon^{*0}|^2 \right\}$$

$$= \frac{1}{5} \int \frac{d\Omega}{4\pi} \left\{ \left| -\tfrac{1}{\sqrt{2}} \cos\theta\, e^{-i\phi} \right|^2 + |-\sin\theta|^2 + \left| -\tfrac{i}{\sqrt{2}} e^{-i\phi} \right|^2 \right\}$$

$$= \frac{1}{5} \int \frac{d\Omega}{4\pi} \left\{ \tfrac{1}{2}\cos^2\theta + \sin^2\theta + \tfrac{1}{2} \right\}$$

$$= \frac{1}{10} \int d\cos\theta \left\{ \tfrac{1}{2}\cos^2\theta + \tfrac{1}{2} + \sin^2\theta \right\}$$

$$= \frac{1}{10}\left(\frac{8}{3}\right)$$

$$= \frac{4}{15} //$$

$$\gamma_{-1} = \sum_{\varepsilon} \int \frac{d\Omega}{4\pi} \left| C_{0-1} \right|^2 + \left| C_{-1-1} \right|^2$$

$$= \sum_{\varepsilon} \int \frac{d\Omega}{4\pi} \left| \sqrt{\tfrac{1}{5}}\, \varepsilon^{*-} \right|^2 + \left| \sqrt{\tfrac{1}{5}}\, \varepsilon^{*0} \right|^2$$

$$= \frac{1}{5} \int \frac{d\Omega}{4\pi} \left\{ \left| \frac{1}{\sqrt{2}} \cos\theta\, e^{-i\phi} \right|^2 + \left| -\sin\theta \right|^2 + \frac{1}{2} \right\}$$

$$= \frac{1}{5} \int \frac{d\Omega}{4\pi} \left\{ \frac{1}{2}\cos^2\theta + \sin^2\theta + \frac{1}{2} \right\}$$

$$= \frac{4}{15} //$$

$$\gamma_0 = \sum_{\varepsilon} \int \frac{d\Omega}{4\pi} \left| C_{00} \right|^2 + \left| C_{10} \right|^2 + \left| C_{-10} \right|^2$$

$$= \sum_{\varepsilon} \int \frac{d\Omega}{4\pi} \left| \sqrt{\tfrac{4}{15}}\, \varepsilon^{*0} \right|^2 + \left| \sqrt{\tfrac{1}{15}}\, \varepsilon^{*-} \right|^2 + \left| \sqrt{\tfrac{1}{15}}\, \varepsilon^{*+} \right|^2$$

$$= \frac{1}{15} \int \frac{d\Omega}{4\pi} \sum_{\varepsilon} \left\{ 4 \left| \varepsilon^{*0} \right|^2 + \left| \varepsilon^{*-} \right|^2 + \left| \varepsilon^{*+} \right|^2 \right\}$$

$$= \frac{1}{15} \int \frac{d\Omega}{4\pi} \left\{ 4\sin^2\theta + \frac{1}{2}\cos^2\theta + \frac{1}{2}\cos^2\theta + \frac{1}{2} + \frac{1}{2} \right\}$$

$$= \frac{1}{15} \frac{1}{2} \int d\cos\theta \left\{ 3\sin^2\theta + 2 \right\}$$

$$= \frac{1}{15} \frac{1}{2}\, 8$$

$$= 4/15 //$$

We find that the decay widths are all the same:

$$\Gamma_{m'} = \Gamma \gamma_{m'}$$

$$= \frac{2\alpha\omega^3}{c^2} \frac{3^4 2^{14} (7!)^2}{5^{15}} a_0^2 \frac{4}{15}$$

$$= \frac{\alpha\omega^3}{c^2} a_0^2 \frac{3^3 2^{17} (7!)^2}{5^{16}}$$

$$\simeq 589 \frac{\alpha\omega^3}{c^2} a_0^2$$

C. When we include fine structure, the eigenstates will be states with total angular momentum $\vec{J} = \hat{L} + \hat{S}$; $|n, J, J_z, l, s\rangle$.

Matrix element of the dipole operator with these states is what we seek.

$$\Delta H^{J' J_z'}_{J J_z} = \langle 2P, J, J_z | \vec{\varepsilon} \cdot \vec{r} | 3D, J', J_z' \rangle$$

To take advantage of earlier computations let's decompose the J states in terms of L & S states as follows.

$$|3D, J', J_z'\rangle = \sum_{m_l'=-2}^{2} \sum_{m_s'=-\frac{1}{2}}^{\frac{1}{2}} |2, m_l'; \tfrac{1}{2}, m_s'\rangle \langle 2, m_l'; \tfrac{1}{2}, m_s' | 3D, J', J_z'\rangle$$

$$\langle 2P, J, J_z| = \sum_{m_l=-1}^{1} \sum_{m_s=-\frac{1}{2}}^{\frac{1}{2}} \langle 2P, J, J_z | 1, m_l; \tfrac{1}{2}, m_s\rangle \langle 1, m_l; \tfrac{1}{2}, m_s|$$

$$\Rightarrow \Delta H_{JJ_z}^{J'J_z'} = \sum_{m_\ell, m_s} \langle 2P, J, J_z | 1, m_\ell ; \tfrac{1}{2}, m_s \rangle \langle 1, m_\ell ; \tfrac{1}{2}, m_s | (\vec{\varepsilon} \cdot \vec{r})$$

$$\times \sum_{m_\ell', m_s'} | 2, m_\ell' ; \tfrac{1}{2}, m_s' \rangle \langle 2, m_\ell' ; \tfrac{1}{2}, m_s' | 3D, J', J_z' \rangle$$

Note the $\vec{\varepsilon} \cdot \vec{r}$ has no spin part, so $\langle 1, m_\ell ; \frac{1}{2}, m_s | \vec{\varepsilon} \cdot \vec{r} | 2, m_\ell' ; \frac{1}{2}, m_s' \rangle$

$$= \langle 1, m_\ell | \vec{\varepsilon} \cdot \vec{r} | 2, m_\ell' \rangle \langle \tfrac{1}{2}, m_s | \tfrac{1}{2}, m_s' \rangle$$

$$= \langle 1, m_\ell | \vec{\varepsilon} \cdot \vec{r} | 2, m_\ell' \rangle \delta_{m_s m_s'} = \delta_{m_s m_s'} \langle n=2, \ell=1, m_\ell | \vec{\varepsilon} \cdot \vec{r} | n=3, \ell=2, m_\ell' \rangle$$

$$\Rightarrow \Delta H_{JJ_z}^{J'J_z'} = \sum_{m_\ell, m_\ell'} \sum_{m_s, m_s'} \Big\{ \delta_{m_s m_s'} \Big( \langle 2P, J, J_z | 1, m_\ell ; \tfrac{1}{2}, m_s \rangle \times \langle 2, m_\ell' ; \tfrac{1}{2}, m_s' | 3D, J', J_z' \rangle \Big) \times \langle 2, 1, m_\ell | \vec{\varepsilon}^* \cdot \vec{r} | 3, 2, m_\ell' \rangle \Big\}$$

$$\Delta H_{JJ_z}^{J'J_z'} = \sum_{m_\ell, m_\ell'} \sum_{m_s} \Big\{ \langle 2P, J, J_z | 1, m_\ell ; \tfrac{1}{2}, m_s \rangle \langle 2, m_\ell' ; \tfrac{1}{2}, m_s | 3D, J', J_z' \rangle \times \underbrace{\langle 2, 1, m_\ell | \vec{\varepsilon}^* \cdot \vec{r} | 3, 2, m_\ell' \rangle}_{= \Delta H_{m_\ell m_\ell'} \text{ from part b}} \Big\}$$

$= N C_{m_\ell m_\ell'}$, where $N = \dfrac{3^2 2^7 7!}{5^{15/2}} a_0$

$$\boxed{\therefore\ \Delta H_{JJ_z}^{J'J_z'} = \sum_{m_\ell, m_\ell'} \sum_{m_s} \left\{ \langle 2P, J, J_z | 1, m_\ell; \tfrac{1}{2}, m_s \rangle \langle 2, m_\ell'; \tfrac{1}{2}, m_s | 3D, J', J_z' \rangle N C_{m_\ell m_\ell'} \right\}}$$

The decay rate of a $(J', J_z')$ state will then be

$$\Gamma^{J'J_z'} = \frac{2\alpha\omega^3}{c^2} \int \frac{d\Omega}{4\pi} \sum_{\varepsilon} \sum_{JJ_z'} \left| \Delta H_{JJ_z}^{J', J_z'} \right|^2$$

Let's calculate the various matrix elements.

$$\Delta H_{JJ_z}^{5/2, 5/2} = \sum_{m_\ell} \langle 2P, J, J_z | 1, m_\ell; \tfrac{1}{2}, \tfrac{1}{2} \rangle \langle 2, 2; \tfrac{1}{2}, \tfrac{1}{2} | 3D, 5/2, 5/2 \rangle N C_{m_\ell 2}$$

$$= \langle 2P, J, J_z | 1, 1; \tfrac{1}{2}, \tfrac{1}{2} \rangle \langle 2, 2; \tfrac{1}{2}, \tfrac{1}{2} | 3D, 5/2, 5/2 \rangle N C_{12}$$

b/c $\quad C_{m_\ell 2} = C_{12}\, \delta_{m_\ell 1}$

$$= \langle 2P, J, J_z | 1, 1; \tfrac{1}{2}, \tfrac{1}{2} \rangle N C_{12}$$

$$\Rightarrow \Delta H_{JJ_z}^{5/2, 5/2} = \begin{cases} N C_{12} & \text{for } J = \tfrac{3}{2},\ J_z = 3/2, \\ 0 & \text{otherwise} \end{cases}$$

Similarly, $\Delta H_{JJ_z}^{5/2, -5/2} = \langle 2P, J, J_z | 1, -1; \tfrac{1}{2}, -\tfrac{1}{2} \rangle N C_{-1-2}$

$$= \begin{cases} N C_{-1-2} & , \text{ for } J = -3/2,\ J_z = -3/2 \\ 0 & \text{otherwise.} \end{cases}$$

$$\Delta H^{5/2,3/2}_{J\,J_z} = \sum \left\{ \langle 2P, J, J_z | 1, m_\ell ; \tfrac{1}{2} m_s \rangle \langle 2, m_\ell' ; \tfrac{1}{2} m_s | 3D, 5/2, 3/2 \rangle N C_{m_\ell m_\ell'} \right\}$$

From here on I'll suppress the $l = 1$, $s = \frac{1}{2}$, 2P, 3D etc. notations.

$$\Delta H^{5/2,3/2}_{JJ_z} = \sum_{m_\ell m_\ell' m_s} \langle J J_z | m_\ell m_s \rangle \langle m_\ell' m_s | 5/2, 3/2 \rangle N C_{m_\ell m_\ell'}$$

$$= \langle J J_z | \sum_{m_\ell} \Big\{ |m_\ell -\tfrac{1}{2}\rangle \langle 2, -\tfrac{1}{2} | \tfrac{5}{2}, 3/2 \rangle N C_{m_\ell 2}$$

$$+ |m_\ell \tfrac{1}{2}\rangle \langle 1 \tfrac{1}{2} | \tfrac{5}{2}, 3/2 \rangle N C_{m_\ell 1} \Big\}$$

$$= \langle J J_z | \sum_{m_\ell} \Big\{ |m_\ell -\tfrac{1}{2}\rangle \sqrt{\tfrac{1}{5}}\, N C_{12}\, \delta_{m_\ell 1}$$

$$+ |m_\ell \tfrac{1}{2}\rangle \sqrt{\tfrac{4}{5}}\, N C_{m_\ell 1} \Big\}$$

$$= \langle J J_z | \Big\{ |1 -\tfrac{1}{2}\rangle \sqrt{\tfrac{1}{5}}\, N C_{12} + |1 \tfrac{1}{2}\rangle \sqrt{\tfrac{4}{5}}\, N C_{11} + |0 \tfrac{1}{2}\rangle \sqrt{\tfrac{4}{5}}\, N C_{01} \Big\}$$

$$\Delta H^{5/2,\frac{3}{2}}_{\frac{1}{2}\,\frac{1}{2}} = \langle \tfrac{1}{2} \tfrac{1}{2} | 1 -\tfrac{1}{2}\rangle \sqrt{\tfrac{1}{5}}\, N C_{12} + \langle \tfrac{1}{2} \tfrac{1}{2} | 0 \tfrac{1}{2}\rangle \sqrt{\tfrac{4}{5}}\, N C_{01}$$

$$= \sqrt{\tfrac{2}{3}} \sqrt{\tfrac{1}{5}}\, N C_{12} - \sqrt{\tfrac{1}{3}} \sqrt{\tfrac{4}{5}}\, N C_{01}$$

$$= N \left( \sqrt{\tfrac{2}{15}} \sqrt{\tfrac{2}{5}}\, \varepsilon^{*+} - \sqrt{\tfrac{4}{15}} \sqrt{\tfrac{1}{5}}\, \varepsilon^{*+} \right) = 0 \,//$$

$$\Delta H^{5/2,3/2}_{\frac{1}{2}\,-\frac{1}{2}} = 0 \,//$$

$$\Delta H^{5/2,3/2}_{\frac{3}{2},\frac{3}{2}} = \sqrt{\tfrac{4}{5}}\, N C_{11} \,//$$

$$\Delta H^{5/2,3/2}_{\frac{3}{2},\frac{1}{2}} = \langle \tfrac{3}{2} \tfrac{1}{2} | 1 -\tfrac{1}{2}\rangle \sqrt{\tfrac{1}{5}}\, N C_{12} + \langle \tfrac{3}{2} \tfrac{1}{2} | 0 \tfrac{1}{2}\rangle \sqrt{\tfrac{4}{5}}\, N C_{01}$$

$$= \sqrt{\tfrac{1}{15}}\, N C_{12} + \sqrt{\tfrac{8}{15}}\, N C_{01} \,//$$

$$\Delta H^{5/2,3/2}_{\frac{3}{2},-\frac{1}{2}} = \Delta H^{5/2,3/2}_{3/2,-3/2} = 0 \,//$$

$$\Delta H^{5/2,\,1/2}_{J\,J_z} = \sum_{m_\ell m_\ell' m_s} \langle J J_z | m_\ell m_s \rangle \langle m_\ell' m_s | 5/2\ \tfrac{1}{2} \rangle N C_{m_\ell m_\ell'}$$

$$= \sum_{m_\ell} \langle J J_z | \Big\{ |m_\ell\ \tfrac{1}{2}\rangle \langle 0\ \tfrac{1}{2} | 5/2\ \tfrac{1}{2} \rangle N C_{m_\ell 0} + |m_\ell\ -\tfrac{1}{2}\rangle \langle 1\ -\tfrac{1}{2} | 5/2\ \tfrac{1}{2} \rangle N C_{m_\ell 1} \Big\}$$

$$= \langle J J_z | \sum_{m_\ell} \Big\{ |m_\ell\ \tfrac{1}{2}\rangle \sqrt{\tfrac{3}{5}} N C_{m_\ell 0} + |m_\ell\ -\tfrac{1}{2}\rangle \sqrt{\tfrac{2}{5}} N C_{m_\ell 1} \Big\}$$

$$= \langle J J_z | \Big\{ \sqrt{\tfrac{3}{5}} N \Big( C_{10} |1\ \tfrac{1}{2}\rangle + C_{00} |0\ \tfrac{1}{2}\rangle + C_{-10} |-1\ \tfrac{1}{2}\rangle \Big) + \sqrt{\tfrac{2}{5}} N \Big( C_{11} |1\ -\tfrac{1}{2}\rangle + C_{01} |0\ -\tfrac{1}{2}\rangle \Big) \Big\}$$

$$\Delta H^{5/2,\,1/2}_{\frac{3}{2},\,\frac{3}{2}} = \sqrt{\tfrac{3}{5}} N C_{10}$$

$$\Delta H^{5/2,\,1/2}_{\frac{3}{2},\,\frac{1}{2}} = \sqrt{\tfrac{3}{5}} N C_{00} \sqrt{\tfrac{2}{3}} + \sqrt{\tfrac{2}{5}} N C_{11} \sqrt{\tfrac{1}{3}} = \sqrt{\tfrac{2}{5}} N C_{00} + \sqrt{\tfrac{2}{15}} N C_{11}$$

$$\Delta H^{5/2,\,1/2}_{\frac{3}{2},\,-\frac{1}{2}} = \sqrt{\tfrac{3}{5}} N C_{-10} \sqrt{\tfrac{1}{3}} + \sqrt{\tfrac{2}{5}} N C_{01} \sqrt{\tfrac{2}{3}} = \sqrt{\tfrac{1}{5}} N C_{-10} + \sqrt{\tfrac{4}{15}} N C_{01}$$

$$\Delta H^{5/2,\,1/2}_{\frac{3}{2},\,-\frac{3}{2}} = 0$$

$$\Delta H^{5/2,\,1/2}_{\frac{1}{2},\,\frac{1}{2}} = \sqrt{\tfrac{3}{5}} N C_{00} \left(-\sqrt{\tfrac{1}{3}}\right) + \sqrt{\tfrac{2}{5}} N C_{11} \sqrt{\tfrac{2}{3}} = N\left(-\sqrt{\tfrac{1}{5}} C_{00} + \sqrt{\tfrac{4}{15}} C_{11}\right)$$

$$= N\left(-\sqrt{\tfrac{1}{5}}\sqrt{\tfrac{4}{15}}\, \varepsilon^{*0} + \sqrt{\tfrac{4}{15}}\sqrt{\tfrac{1}{5}}\, \varepsilon^{*0}\right)$$

$$= 0$$

$$\Delta H^{5/2,\,1/2}_{\frac{1}{2},\,-\frac{1}{2}} = \sqrt{\tfrac{3}{5}} N C_{-10} \left(-\sqrt{\tfrac{2}{3}}\right) + \sqrt{\tfrac{2}{5}} N C_{01} \sqrt{\tfrac{1}{3}} = N\left(-\sqrt{\tfrac{2}{5}} C_{-10} + \sqrt{\tfrac{2}{15}} C_{01}\right)$$

$$= N\left(-\sqrt{\tfrac{2}{5}}\sqrt{\tfrac{1}{15}}\, \varepsilon^{*+} + \sqrt{\tfrac{2}{15}}\sqrt{\tfrac{1}{5}}\, \varepsilon^{*+}\right) = 0$$

$$\Delta H^{5/2\,-3/2}_{J\,J_z} = \sum \langle J J_z | m_\ell m_s \rangle \langle m_\ell' m_s | 5/2\ -\tfrac{3}{2} \rangle N C_{m_\ell m_\ell'}$$

$$= \langle J J_z | \sum_{m_\ell} \Big\{ |m_\ell + \tfrac{1}{2}\rangle \langle -2 + \tfrac{1}{2} | 5/2\ -\tfrac{3}{2} \rangle N C_{m_\ell -2}$$

$$+ |m_\ell - \tfrac{1}{2}\rangle \langle -1 - \tfrac{1}{2} | 5/2\ -3/2 \rangle N C_{m_\ell -1} \Big\}$$

$$= \langle J J_z | \Big\{ |-1 + \tfrac{1}{2}\rangle \sqrt{\tfrac{1}{5}}\, N\, C_{-1-2} + |0 - \tfrac{1}{2}\rangle \sqrt{\tfrac{4}{5}}\, N C_{0-1}$$

$$+ |-1 - \tfrac{1}{2}\rangle \sqrt{\tfrac{4}{5}}\, N C_{-1-1} \Big\}$$

$$= \langle J J_z | \Big\{ |-1 + \tfrac{1}{2}\rangle \sqrt{\tfrac{1}{5}}\, N C_{-1-2} + \sqrt{\tfrac{4}{5}}\, N \Big( |0 - \tfrac{1}{2}\rangle C_{0-1} + |-1 - \tfrac{1}{2}\rangle C_{-1-1} \Big) \Big\}$$

$$\Delta H^{5/2\,-\frac{3}{2}}_{\frac{3}{2}\,\frac{3}{2}} = \Delta H^{5/2\,-3/2}_{3/2\,\frac{1}{2}} = 0$$

$$\Delta H^{5/2\,-\frac{3}{2}}_{\frac{3}{2}\,-\frac{1}{2}} = \sqrt{\tfrac{1}{15}}\, N\, C_{-1-2} + \sqrt{\tfrac{8}{15}}\, N\, C_{0-1} \ //$$

$$\Delta H^{5/2\,-\frac{3}{2}}_{\frac{3}{2}\,-\frac{3}{2}} = \sqrt{\tfrac{4}{5}}\, N\, C_{-1-1} \ //$$

$$\Delta H^{5/2\,-\frac{3}{2}}_{\frac{1}{2}\,\frac{1}{2}} = 0 \ //$$

$$\Delta H^{5/2\,-3/2}_{\frac{1}{2}\,-\frac{1}{2}} = -\sqrt{\tfrac{2}{3}}\sqrt{\tfrac{1}{5}}\, N C_{-1-2} + \sqrt{\tfrac{4}{5}}\sqrt{\tfrac{1}{3}}\, N\, C_{0-1}$$

$$= N\Big( -\sqrt{\tfrac{2}{15}}\sqrt{\tfrac{2}{5}}\, \varepsilon^{*-} + \sqrt{\tfrac{4}{15}}\sqrt{\tfrac{1}{5}}\, \varepsilon^{*-} \Big) = 0 \ //$$

Onto transitions to $J' = 3/2$:

$$\Delta H^{3/2\,3/2}_{J\,J_z} = \sum \langle J J_z | m_\ell m_s \rangle \langle m_\ell' m_s | \tfrac{3}{2}\,\tfrac{3}{2} \rangle N C_{m_\ell m_\ell'}$$

$$= \langle J J_z | \sum_{m_\ell} \Big\{ |m_\ell -\tfrac{1}{2}\rangle \langle 2 -\tfrac{1}{2} | \tfrac{3}{2} \tfrac{3}{2}\rangle N C_{m_\ell 2}$$

$$+ |m_\ell \tfrac{1}{2}\rangle \langle 1 \tfrac{1}{2} | \tfrac{3}{2} \tfrac{3}{2}\rangle N C_{m_\ell 1} \Big\}$$

$$= \langle J J_z | \Big\{ |1 -\tfrac{1}{2}\rangle \sqrt{\tfrac{4}{5}} N C_{12} - \sqrt{\tfrac{1}{5}} N \big( |0 \tfrac{1}{2}\rangle C_{01} + |1 \tfrac{1}{2}\rangle C_{11} \big) \Big\}$$

$$\Delta H^{\frac{3}{2}\frac{3}{2}}_{\frac{3}{2}\frac{3}{2}} = -\sqrt{\tfrac{1}{5}} N C_{11} //$$

$$\Delta H^{\frac{3}{2}\frac{3}{2}}_{\frac{3}{2}\frac{1}{2}} = \sqrt{\tfrac{1}{3}}\sqrt{\tfrac{4}{5}} N C_{12} - \sqrt{\tfrac{1}{5}}\sqrt{\tfrac{2}{3}} N C_{01} = \sqrt{\tfrac{4}{15}} N C_{12} - \sqrt{\tfrac{2}{15}} N C_{01} //$$

$$\Delta H^{3/2\, 3/2}_{\frac{3}{2}\, -\frac{1}{2}} = \Delta H^{3/2\, 3/2}_{3/2\, -3/2} = 0 //$$

$$\Delta H^{3/2\, 3/2}_{\frac{1}{2}\frac{1}{2}} = \sqrt{\tfrac{2}{3}}\sqrt{\tfrac{4}{5}} N C_{12} + \sqrt{\tfrac{1}{5}}\sqrt{\tfrac{1}{3}} N C_{01} //$$

$$\Delta H^{3/2\, 3/2}_{\frac{1}{2}\, -\frac{1}{2}} = 0 //$$

$$\Delta H^{3/2\, \frac{1}{2}}_{J J_z} = \langle J J_z | \sum_{m_\ell} \Big\{ |m_\ell -\tfrac{1}{2}\rangle \langle 1 -\tfrac{1}{2} | \tfrac{3}{2} \tfrac{1}{2}\rangle N C_{m_\ell 1}$$

$$+ |m_\ell \tfrac{1}{2}\rangle \langle 0 \tfrac{1}{2} | \tfrac{3}{2} \tfrac{1}{2}\rangle N C_{m_\ell 0} \Big\}$$

$$= \langle J J_z | \sum_{m_\ell} \Big\{ |m_\ell -\tfrac{1}{2}\rangle \sqrt{\tfrac{3}{5}} N C_{m_\ell 1} + |m_\ell \tfrac{1}{2}\rangle \Big(-\sqrt{\tfrac{2}{5}}\Big) N C_{m_\ell 0} \Big\}$$

$$= \langle J J_z | \Big\{ \sqrt{\tfrac{3}{5}} N \big( |1 -\tfrac{1}{2}\rangle C_{11} + |0 -\tfrac{1}{2}\rangle C_{01} \big)$$

$$- \sqrt{\tfrac{2}{5}} N \big( |-1 \tfrac{1}{2}\rangle C_{-10} + |0 \tfrac{1}{2}\rangle C_{00} + |1 \tfrac{1}{2}\rangle C_{10} \big) \Big\}$$

$$\Rightarrow \Delta H^{3/2\, \frac{1}{2}}_{\frac{3}{2}\frac{3}{2}} = -\sqrt{\tfrac{2}{5}} N C_{10} //$$

$$\Rightarrow \Delta H^{\frac{3}{2}\frac{1}{2}}_{\frac{3}{2}\frac{1}{2}} = \sqrt{\tfrac{3}{5}}N\sqrt{\tfrac{1}{3}}C_{11} - \sqrt{\tfrac{2}{5}}N\sqrt{\tfrac{2}{3}}C_{00} = \sqrt{\tfrac{1}{15}}NC_{11} - \sqrt{\tfrac{4}{15}}NC_{00}$$

$$\Delta H^{3/2\,\frac{1}{2}}_{\frac{3}{2}\,-\frac{1}{2}} = \sqrt{\tfrac{3}{5}}N\sqrt{\tfrac{2}{3}}C_{01} - \sqrt{\tfrac{2}{5}}N\sqrt{\tfrac{1}{3}}C_{-10} = \sqrt{\tfrac{2}{5}}NC_{01} - \sqrt{\tfrac{2}{15}}NC_{-10}$$

$$\Delta H^{3/2\,\frac{1}{2}}_{\frac{3}{2}\,-\frac{3}{2}} = 0$$

$$\Delta H^{3/2\,\frac{1}{2}}_{\frac{1}{2}\,\frac{1}{2}} = \sqrt{\tfrac{3}{5}}N\sqrt{\tfrac{2}{3}}C_{11} + \sqrt{\tfrac{2}{5}}N\sqrt{\tfrac{1}{3}}C_{00} = \sqrt{\tfrac{2}{5}}NC_{11} + \sqrt{\tfrac{2}{15}}NC_{00}$$

$$\Delta H^{3/2\,1/2}_{\frac{1}{2}\,-\frac{1}{2}} = \sqrt{\tfrac{3}{5}}N\sqrt{\tfrac{1}{3}}C_{01} + \sqrt{\tfrac{2}{5}}N\sqrt{\tfrac{2}{3}}C_{-10} = \sqrt{\tfrac{1}{5}}NC_{01} + \sqrt{\tfrac{4}{15}}NC_{-10}$$

$$\Delta H^{\frac{3}{2}\,-\frac{1}{2}}_{J\,J_z} = \langle J J_z | \sum_{m_\ell} \{ |m_\ell\, -\tfrac{1}{2}\rangle\langle 0\, -\tfrac{1}{2} | \tfrac{3}{2}\, -\tfrac{1}{2}\rangle N C_{m_\ell 0}$$
$$+ |m_\ell\, \tfrac{1}{2}\rangle\langle -1\, +\tfrac{1}{2} | \tfrac{3}{2}\, -\tfrac{1}{2}\rangle N C_{m_\ell -1} \}$$

$$= \langle J J_z | \sum_{m_\ell} \{ |m_\ell\, -\tfrac{1}{2}\rangle \sqrt{\tfrac{2}{5}} N C_{m_\ell 0} + |m_\ell\, \tfrac{1}{2}\rangle \sqrt{\tfrac{3}{5}} N C_{m_\ell -1} \}$$

$$= \langle J J_z | \{ \sqrt{\tfrac{2}{5}}N\left( |-1\, -\tfrac{1}{2}\rangle C_{-10} + |0\, -\tfrac{1}{2}\rangle C_{00} + |1\, -\tfrac{1}{2}\rangle C_{10} \right)$$
$$- \sqrt{\tfrac{3}{5}}N\left( |-1\, \tfrac{1}{2}\rangle C_{-1-1} + |0\, \tfrac{1}{2}\rangle C_{0-1} \right) \}$$

$$\Rightarrow \Delta H^{\frac{3}{2}\,-\frac{1}{2}}_{\frac{3}{2}\,-\frac{3}{2}} = \sqrt{\tfrac{2}{5}}NC_{-10}$$

$$\Delta H^{3/2\,-\frac{1}{2}}_{\frac{3}{2}\,-\frac{1}{2}} = \sqrt{\tfrac{2}{5}}N\sqrt{\tfrac{2}{3}}C_{00} - \sqrt{\tfrac{3}{5}}N\sqrt{\tfrac{1}{3}}C_{-1-1} = \sqrt{\tfrac{4}{15}}NC_{00} - \sqrt{\tfrac{1}{5}}NC_{-1-1}$$

$$\Delta H^{3/2\,-\frac{1}{2}}_{\frac{3}{2}\,\frac{1}{2}} = \sqrt{\tfrac{2}{5}}N\sqrt{\tfrac{1}{3}}C_{10} - \sqrt{\tfrac{3}{5}}N\sqrt{\tfrac{2}{3}}C_{0-1} = \sqrt{\tfrac{2}{15}}NC_{10} - \sqrt{\tfrac{2}{5}}NC_{0-1}$$

$$\Delta H^{3/2\,-\frac{1}{2}}_{3/2\,3/2} = 0$$

$$\Delta H^{3/2\,-\frac{1}{2}}_{\frac{1}{2}\,-\frac{1}{2}} = \sqrt{\tfrac{2}{5}}N\sqrt{\tfrac{1}{3}}C_{00} + \sqrt{\tfrac{3}{5}}N\sqrt{\tfrac{2}{3}}C_{-1-1} = \sqrt{\tfrac{2}{15}}NC_{00} + \sqrt{\tfrac{2}{5}}NC_{-1-1}$$

$$\Delta H^{3/2\,-\frac{1}{2}}_{\frac{1}{2}\,\frac{1}{2}} = \sqrt{\tfrac{2}{5}}N\sqrt{\tfrac{2}{3}}C_{10} + \sqrt{\tfrac{3}{5}}N\sqrt{\tfrac{1}{3}}C_{0-1} = \sqrt{\tfrac{4}{15}}NC_{10} + \sqrt{\tfrac{1}{5}}NC_{0-1}$$

$$\Delta H^{3/2\,-3/2}_{J\,J_z} = \langle J J_z | \sum_{m_\ell} \Big\{ |m_\ell\,-\tfrac{1}{2}\rangle \langle -1\,-\tfrac{1}{2}|\tfrac{3}{2}\,-\tfrac{3}{2}\rangle N C_{m_\ell -1}$$

$$+ |m_\ell\,\tfrac{1}{2}\rangle\langle -2\,\tfrac{1}{2}|\tfrac{3}{2}\,-\tfrac{3}{2}\rangle N C_{m_\ell -2}\Big\}$$

$$= \langle J J_z|\sum_{m_\ell}\Big\{ |m_\ell\,-\tfrac{1}{2}\rangle\sqrt{\tfrac{1}{5}}NC_{m_\ell-1} - |m_\ell\,\tfrac{1}{2}\rangle\sqrt{\tfrac{4}{5}}NC_{m_\ell-2}\Big\}$$

$$= \langle J J_z|\Big\{\sqrt{\tfrac{1}{5}}N\big(|-1\,-\tfrac{1}{2}\rangle C_{-1-1} + |0\,-\tfrac{1}{2}\rangle C_{0-1}\big)$$

$$-\sqrt{\tfrac{4}{5}}N\,|-1\,\tfrac{1}{2}\rangle C_{-1-2}\Big\}$$

$$\Rightarrow \Delta H^{\frac{3}{2}-\frac{3}{2}}_{\frac{3}{2}\,\frac{3}{2}} = \Delta H^{\frac{3}{2}-\frac{3}{2}}_{\frac{3}{2}\,\frac{1}{2}} = 0$$

$$\Delta H^{\frac{3}{2}-\frac{3}{2}}_{\frac{3}{2}\,-\frac{1}{2}} = \sqrt{\tfrac{1}{5}}N\sqrt{\tfrac{2}{3}}C_{0-1} - \sqrt{\tfrac{4}{5}}N\sqrt{\tfrac{1}{3}}C_{-1-2} = \sqrt{\tfrac{2}{15}}NC_{0-1} - \sqrt{\tfrac{4}{15}}NC_{-1-2}$$

$$\Delta H^{\frac{3}{2}-\frac{3}{2}}_{\frac{3}{2}\,-\frac{3}{2}} = \sqrt{\tfrac{1}{5}}NC_{-1-1}$$

$$\Delta H^{\frac{3}{2}-\frac{3}{2}}_{\frac{1}{2}\,\frac{1}{2}} = 0$$

$$\Delta H^{3/2\,-3/2}_{\frac{1}{2}\,-\frac{1}{2}} = \sqrt{\tfrac{1}{5}}N\sqrt{\tfrac{1}{3}}C_{0-1} + \sqrt{\tfrac{4}{5}}N\sqrt{\tfrac{2}{3}}C_{-1-2} = \sqrt{\tfrac{1}{15}}NC_{0-1} + \sqrt{\tfrac{8}{15}}NC_{-1-2}$$

As we can see from these results there are no matrix elements between $J'=\frac{5}{2}$ and $J=\frac{1}{2}$ states.

Decay widths

$$\Gamma^{J' J_z'} = \frac{2\alpha\omega^3}{c^2}\int\frac{d\Omega}{4\pi}\sum_{\varepsilon}\sum_{J, J_z}\left|\Delta H^{J', J_z'}_{J\, J_z}\right|^2$$

Here, we've ignored the correction to the energies of the photon due to fine structure splitting.

$$= \frac{2\alpha\omega^3}{c^2}N^2\int\frac{d\Omega}{4\pi}\sum_{\varepsilon}\sum_{J, J_z}\left|\frac{1}{N}\Delta H^{J' J_z'}_{J\, J_z}\right|^2$$

$$= \Gamma\int\frac{d\Omega}{4\pi}\sum_{\varepsilon}\sum_{J, J_z}\left|\frac{1}{N}\Delta H^{J' J_z'}_{J\, J_z}\right|^2$$

$$= \Gamma\ \gamma^{J' J_z'}$$

$$\gamma^{5/2\ 5/2} = \int\frac{d\Omega}{4\pi}\sum_{\varepsilon}|C_{12}|^2$$

$= \gamma = \frac{4}{15}$ as we found in (1b).

$\Rightarrow \Gamma^{5/2\ 5/2} = \Gamma\gamma$ = the same as in (1b). Similarly $\Gamma^{5/2\,-5/2} = \Gamma\gamma$ by rotational symmetry.

For $J_z' = \frac{3}{2}$; $\displaystyle\sum_{J_z}\left|\frac{1}{N}\Delta H^{5/2,\ 3/2}_{3/2\ \ J_z}\right|^2 = \left|\sqrt{\tfrac{4}{5}}\ C_{11}\right|^2 + \left|\sqrt{\tfrac{1}{15}}\ C_{12} + \sqrt{\tfrac{8}{15}}\ C_{01}\right|^2$

$$= \left|\frac{2}{5}\ \varepsilon^{*0}\right|^2 + \left|\frac{1}{5}\sqrt{\frac{2}{3}}\ \varepsilon^{*+} + \frac{2}{5}\sqrt{\frac{2}{3}}\ \varepsilon^{*+}\right|^2$$

$$= \left|\frac{2}{5}\ \varepsilon^{*0}\right|^2 + \left|\frac{3}{5}\sqrt{\frac{2}{3}}\ \varepsilon^{*+}\right|^2$$

$$\sum_{\varepsilon} \rightarrow \quad \frac{4}{25}\left(\sin^2\theta\right) + \frac{6}{25}\left(\frac{1}{2}\cos^2\theta + \frac{1}{2}\right)$$

$$\Rightarrow \quad \gamma^{\frac{5}{2},\frac{3}{2}} = \frac{1}{25}\left(\frac{1}{2}\right)\int d\cos\theta \left(4\sin^2\theta + 3\cos^2\theta + 3\right)$$

$$= \frac{1}{25}\left(\frac{1}{2}\right)\frac{40}{3}$$

$$= \frac{4}{15}$$

$$\cdot \quad \gamma^{\frac{5}{2}\frac{1}{2}} = \int \frac{d\Omega}{4\pi} \sum_{\varepsilon} \left\{ \left| \sqrt{\frac{3}{5}}\, C_{10} \right|^2 + \left| \sqrt{\frac{2}{5}}\, C_{00} + \sqrt{\frac{2}{15}}\, C_{11} \right|^2 \right.$$

$$\left. + \left| \sqrt{\frac{1}{5}}\, C_{-10} + \sqrt{\frac{4}{15}}\, C_{01} \right|^2 \right\}$$

$$= \int \frac{d\Omega}{4\pi} \sum_{\varepsilon} \left\{ \frac{1}{25}|\varepsilon^{*-}|^2 + \left| \frac{2}{5}\sqrt{\frac{2}{3}}\, \varepsilon^{*0} + \frac{1}{5}\sqrt{\frac{2}{3}}\, \varepsilon^{*0} \right|^2 \right.$$

$$\left. + \left| \frac{1}{5}\sqrt{\frac{1}{3}}\, \varepsilon^{*+} + \frac{2}{5}\sqrt{\frac{1}{3}}\, \varepsilon^{*+} \right|^2 \right\}$$

$$= \frac{1}{25}\int \frac{d\Omega}{4\pi} \sum_{\varepsilon} \left\{ |\varepsilon^{*-}|^2 + 6|\varepsilon^{*0}|^2 + 3|\varepsilon^{*+}|^2 \right\}$$

$$= \frac{1}{25}\int \frac{d\Omega}{4\pi} \left\{ \frac{1}{2}\cos^2\theta + 6\sin^2\theta + \frac{3}{2}\cos^2\theta + \frac{1}{2} + \frac{3}{2} \right\}$$

$$= \frac{1}{25}\left(\frac{1}{2}\right)\int d\cos\theta \left\{ 4 + 4\sin^2\theta \right\}$$

$$= \frac{1}{25}\,\frac{1}{2}\,\frac{40}{3}$$

$$= \frac{4}{15}$$

Let's show that the result holds to a few of the $J' = \frac{3}{2}$ states

$$\gamma^{\frac{3}{2}\frac{3}{2}} = \int \frac{d\Omega}{4\pi} \sum_{\varepsilon} \left\{ \left| -\sqrt{\tfrac{1}{5}}\, C_{11} \right|^2 + \left| \sqrt{\tfrac{4}{15}}\, C_{12} - \sqrt{\tfrac{2}{15}}\, C_{01} \right|^2 + \left| \sqrt{\tfrac{8}{15}}\, C_{12} + \sqrt{\tfrac{1}{15}}\, C_{01} \right|^2 \right\}$$

$$= \int \frac{d\Omega}{4\pi} \sum_{\varepsilon} \left\{ \frac{1}{25} |\varepsilon^0|^2 + \left| \frac{2}{5}\sqrt{\tfrac{2}{3}}\, \varepsilon^{*+} - \frac{1}{5}\sqrt{\tfrac{2}{3}}\, \varepsilon^{*+} \right|^2 + \left| \frac{4}{5}\sqrt{\tfrac{1}{3}}\, \varepsilon^{*+} + \frac{1}{5}\sqrt{\tfrac{1}{3}}\, \varepsilon^{*+} \right|^2 \right\}$$

$$= \frac{1}{25} \int \frac{d\Omega}{4\pi} \sum_{\varepsilon} \left\{ |\varepsilon^0|^2 + \frac{2}{3} |\varepsilon^{*+}|^2 + \frac{25}{3} |\varepsilon^{*+}|^2 \right\}$$

$$= \frac{1}{25} \int \frac{d\Omega}{4\pi} \left\{ \sin^2\theta + \frac{9}{2}\cos^2\theta + \frac{9}{2} \right\}$$

$$= \frac{1}{25}\, \frac{1}{2}\, \frac{40}{3}$$

$$= \frac{4}{15} \quad //$$

$$\gamma^{\frac{3}{2}\frac{1}{2}} = \int \frac{d\Omega}{4\pi} \sum_{\varepsilon} \left\{ \left| -\sqrt{\tfrac{2}{5}}\, C_{10} \right|^2 + \left| \sqrt{\tfrac{1}{5}}\, C_{11} - \sqrt{\tfrac{4}{15}}\, C_{00} \right|^2 + \left| \sqrt{\tfrac{2}{5}}\, C_{01} - \sqrt{\tfrac{2}{15}}\, C_{10} \right|^2 \right.$$
$$\left. + \left| \sqrt{\tfrac{2}{5}}\, C_{11} + \sqrt{\tfrac{2}{15}}\, C_{00} \right|^2 + \left| \sqrt{\tfrac{1}{5}}\, C_{01} + \sqrt{\tfrac{4}{15}}\, C_{-10} \right|^2 \right\}$$

$$= \int \frac{d\Omega}{4\pi} \sum_{\varepsilon} \left\{ \left| -\frac{1}{5}\sqrt{\tfrac{2}{3}}\, \varepsilon^{*-} \right|^2 + \left| \frac{1}{5}\varepsilon^{*0} - \frac{4}{15}\varepsilon^{*0} \right|^2 + \left| \frac{1}{5}\sqrt{2}\, \varepsilon^{*+} - \frac{1}{15}\sqrt{2}\, \varepsilon^{*+} \right|^2 \right.$$
$$\left. + \left| \frac{1}{5}\sqrt{2}\, \varepsilon^{*0} + \frac{2}{15}\sqrt{2}\, \varepsilon^{*0} \right|^2 + \left| \frac{1}{5}\varepsilon^{*+} + \frac{2}{15}\varepsilon^{*+} \right|^2 \right\}$$

$$= \int \frac{d\Omega}{4\pi} \sum_{\varepsilon} \frac{1}{25} \left\{ \frac{2}{3} |\varepsilon^{*-}|^2 + \frac{1}{9} |\varepsilon^{*0}|^2 + \frac{8}{9} |\varepsilon^{*+}|^2 + \frac{50}{9} |\varepsilon^{*0}|^2 + \frac{25}{9} |\varepsilon^{*+}|^2 \right\}$$

$$= \frac{1}{25} \int \frac{d\Omega}{4\pi}\, \frac{1}{9} \sum_{\varepsilon} \left\{ 6 |\varepsilon^{*-}|^2 + 51 |\varepsilon^{*0}|^2 + 33 |\varepsilon^{*+}|^2 \right\}$$

2. (a)

$$\Delta H_m = \langle 2P, m | \vec{\varepsilon}^* \cdot \vec{r} | 3S \rangle$$

$$= \int r^3 dr\, R_{21}^* R_{30} \int d\Omega\, Y_{1m}^* Y_{00}\, \vec{\varepsilon}^* \cdot \hat{r}$$

$$= \int r^3 dr\, R_{21}^* R_{30}\, \vec{\varepsilon}^{*-k} \sqrt{\tfrac{1}{4\pi}} \int d\Omega\, Y_{1m}^* Y_{1j} K_k{}^j$$

|| where $K_k{}^j = \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}_k^{\;j}$, and sum over repeated indices is understood.

$$\varepsilon^{*-k} K_k{}^j Y_{1j} = -\varepsilon^{*-1} Y_{11} - \varepsilon^{*1} Y_{1-1} + \varepsilon^{*0} Y_{10}. \;||$$

$$= \int r^3 dr\, R_{21}^* R_{30} \sqrt{\tfrac{1}{4\pi}}\, \varepsilon^{*-k} K_k{}^j \delta_{mj}$$

$$= \int r^3 dr\, R_{21}^* R_{30} \sqrt{\tfrac{1}{4\pi}}\, \varepsilon^{*-k} K_k{}^m$$

$$|\Delta H_m|^2 = \left| \int r^3 dr\, R_{21}^* R_{30} \right|^2 \frac{1}{4\pi} \left| \varepsilon^{*-k} K_k{}^m \right|^2$$

$$\sum_m |\Delta H_m|^2 = \left| \int r^3 dr\, R_{21}^* R_{30} \right|^2 \frac{1}{4\pi} \varepsilon^{*-k} K_k{}^m \varepsilon^{-l} K_l{}^m . \quad \boxed{\text{b/c } (\varepsilon^k)^* = (-1)^k \varepsilon^{-k}}$$

|| Note that $K_k{}^m K^m{}_l = \sum_m K_k{}^m K^m{}_l = \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}_{kl}$

$$= \delta_{kl}$$

$$= \frac{1}{4\pi} \left| \int dr\, r^3 R_{21}^* R_{30} \right|^2 \varepsilon^{*-k} \delta_{kl} \varepsilon^{-l}$$

$$= \frac{1}{4\pi} \left| \int dr\, r^3 R_{21}^* R_{30} \right|^2 \varepsilon^{*-k} \varepsilon^{-k}, \quad \boxed{\varepsilon^{*-k} \varepsilon^{-k} = \vec{\varepsilon} \cdot \vec{\varepsilon}\,!}$$

$$\sum_{\varepsilon} \varepsilon^{*-k}\varepsilon^{-k} = \varepsilon_1^{*-k}\varepsilon_1^{-k} + \varepsilon_2^{*-k}\varepsilon_2^{-k}$$

$$= \sin^2\theta + \tfrac{1}{2}\cos^2\theta + \tfrac{1}{2}\cos^2\theta + \tfrac{1}{2} + \tfrac{1}{2}$$

$$= \sin^2\theta + \cos^2\theta + 1$$

$$= 2$$

$$\Rightarrow \sum_{\varepsilon}\sum_{m} |\Delta H_m|^2 = \frac{1}{2\pi}\left|\int dr\, r^3 R_{21}^* R_{30}\right|^2$$

$$= \frac{d\Gamma}{d\phi\, d\cos\theta}$$ , independent of angles.

$\Rightarrow$ Angular distribution is isotropic!

(b) Here we have to do 2nd order perturbation theory. We've seen in class that in 2nd order time dependent perturbation theory, the rate of transition from an initial state $|I\rangle$ to final state $|F\rangle$ is

$$\frac{\text{transition}}{\text{sec}} = 2\pi\,\delta(E_F - E_I)\,|a|^2, \quad \text{where } a \text{ is}$$

the amplitude,

$$a = -i\sum_k \frac{\langle F|\Delta H|k\rangle\langle k|\Delta H|I\rangle}{E_I - E_k}$$

In this case

$$a = -i \sum_k \frac{\langle 1S + 2\gamma | \Delta H | k + \gamma \rangle \langle k + \gamma | \Delta H | 3S \rangle}{E(3S) - E(k) - E(\gamma)}$$

Here, the only possible intermediate states have a photon; otherwise we won't get to the 1S - state. Therefore $|k\rangle$ are 2P states!

So, $$a = -i \sum_m \frac{\langle 1S; \gamma_1, \gamma_2 | \Delta H | 2P, m; \gamma_1 \rangle \langle 2P, m; \gamma_1 | \Delta H | 3S \rangle}{E(3S) - E(2P) - E(\gamma_1)}$$

Using the prescription $\langle a + \gamma | \Delta H | b \rangle = q \left( \frac{\omega}{2\epsilon_0} \right)^{\frac{1}{2}} \langle a | \vec{\varepsilon}^{\,*} \cdot \vec{r} | b \rangle$,

$$a = -i \sum_m \frac{q \left( \frac{\omega_2}{2\epsilon_0} \right)^{\frac{1}{2}} \langle 1S | \vec{\varepsilon}^{\,*} \cdot \vec{r} | 2P, m \rangle \, q \left( \frac{\omega_1}{2\epsilon_0} \right)^{\frac{1}{2}} \langle 2P, m | \vec{\varepsilon}^{\,*} \cdot \vec{r} | 3S \rangle}{E(3S) - E(2P) - E(\gamma_1)}$$

$$= -i \frac{q^2}{2\epsilon_0} \sum_m \sqrt{\omega_1 \omega_2} \frac{\langle 1S | \vec{\varepsilon}_2^{\,*} \cdot \vec{r} | 2P, m \rangle \langle 2P, m | \vec{\varepsilon}_1^{\,*} \cdot \vec{r} | 3S \rangle}{E(3S) - E(2P) - \omega_1}$$

Ignoring the correction to photon energy from fine-structure splitting,

$$a = \frac{-i q^2 \sqrt{\omega_1 \omega_2}}{2\epsilon_0} \frac{1}{E(3S) - E(2P) - \omega_1} \sum_m \langle 1S | \vec{\varepsilon}_2^{\,*} \cdot \vec{r} | 2P, m \rangle \langle 2P, m | \vec{\varepsilon}_1^{\,*} \cdot \vec{r} | 3S \rangle$$

$$\sum_m \langle 1s | \vec{\varepsilon}_2^{\,*}\cdot\vec{r} | 2P, m\rangle \langle 2P, m | \vec{\varepsilon}_1^{\,*}\cdot\vec{r} | 3S\rangle$$

$$= \sum_m \left(\int r^3 dr\, R_{10}^* R_{21}\right)\left(\int r^3 dr\, R_{21}^* R_{30}\right)\left(\int d\Omega\, Y_{00}^*(\vec{\varepsilon}_2^{\,*}\cdot\hat{r})\, Y_{2m}\right)$$
$$\left(\int d\Omega\, Y_{2m}^* \,(\vec{\varepsilon}_1^{\,*}\cdot\hat{r})\, Y_{00}\right)$$

$$= \frac{RR'}{4\pi} \sum_m \varepsilon_2^{*-k} K_k^m \, \varepsilon_1^{*-l} K_l^m$$ , where $R, R'$ are the radial integrals.

$$= \frac{RR'}{4\pi} \varepsilon_2^{*\,-k} \delta_{kl}\, \varepsilon_1^{*\,-l}$$

$$= \frac{RR'}{4\pi} \vec{\varepsilon}_2^{\,*} \cdot \vec{\varepsilon}_1^{\,*} ,$$

Here $\vec{\varepsilon}_1$ & $\vec{\varepsilon}_2$ stand for polarization vectors of photons 1 & 2; not the two polarization states of the same photon.

$$\boxed{\therefore\ |a|^2 = \frac{q^4}{4\epsilon_0^2}\, \omega_1\omega_2 \left(\frac{1}{E(3S)-E(2P)-\omega_1}\right)^2 \frac{R^2 R'^2}{(4\pi)^2} \left| \vec{\varepsilon}_2 \cdot \vec{\varepsilon}_1 \right|^2}$$

If the first photon is emitted along the z-axis,

$$\vec{k}_1 = k_1 (0, 0, 1) \Rightarrow \vec{\varepsilon}_1^{(1)} = (1, 0, 0), \quad \vec{\varepsilon}_1^{(2)} = (0, 1, 0).$$

Let the second photon momentum be

$$\vec{k}_2 = k_2 (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$\Rightarrow \vec{\varepsilon}_2^{(1)} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta),$$

$$\vec{\varepsilon}_2^{(2)} = (\sin\phi, -\cos\phi, 0)$$

The sum over all final state polarizations give,

$$\sum_{\alpha_1,\alpha_2} \left| \vec{\varepsilon}_2^{(\alpha_2)} \cdot \vec{\varepsilon}_1^{(\alpha_1)} \right|^2$$

$$= \left|\vec{\varepsilon}_2^{(1)} \cdot \vec{\varepsilon}_1^{(1)}\right|^2 + \left|\vec{\varepsilon}_2^{(1)} \cdot \vec{\varepsilon}_1^{(2)}\right|^2 + \left|\vec{\varepsilon}_2^{(2)} \cdot \vec{\varepsilon}_1^{(1)}\right|^2 + \left|\vec{\varepsilon}_2^{(2)} \cdot \vec{\varepsilon}_1^{(2)}\right|^2$$

$$= \cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\phi + \cos^2\phi$$

$$= \cos^2\theta + 1$$

$$\Rightarrow \sum_{\text{polarizations}} |a|^2 = \frac{q^4}{4\epsilon_0^2}\,\omega_1\omega_2 \left(\frac{1}{E(3S)-E(2P)-\omega_1}\right)^2 \left(\frac{RR'}{4\pi}\right)^2 (\cos^2\theta+1)$$

$$= \left(\frac{q^2}{4\pi\epsilon_0\hbar c}\right)^2 \frac{(\hbar\omega_1)(\hbar\omega_2)\,\pi^2c^2}{\left(-\frac{R_y}{9}+\frac{R_y}{4}-\hbar\omega_1\right)^2} \left(\frac{RR'}{4\pi}\right)^2 (\cos^2\theta+1)$$

$$= \alpha^2\pi^2c^2\,\frac{(\hbar\omega_1)(\hbar\omega_2)}{\left(\frac{5}{36}R_y-\hbar\omega_1\right)^2} \left(\frac{RR'}{4\pi}\right)^2 \left[\cos^2\theta+1\right]$$

The distribution has $(\cos^2\theta+1)$ dependence on angle.

(c) $\langle 2P, J, J_z | \vec{\varepsilon}^* \cdot \vec{r} | 3S, j=\frac{1}{2}, j^3 = +\frac{1}{2} \rangle = \Delta H_{J J_z}$

$$= \sum_{m_l, m_s} \langle 2P, J, J_z | l=1, m_l; s=\tfrac{1}{2}, m_s \rangle \langle l=1, m_l; s=\tfrac{1}{2}, m_s | \vec{\varepsilon}^* \cdot \vec{r} | 3S, j=\tfrac{1}{2}, j^3=+\tfrac{1}{2} \rangle$$

$$= \sum_{m_l, m_s} (\delta_{m_s, +\frac{1}{2}}) \langle 2P, J, J_z | l=1, m_l; s=\tfrac{1}{2}, m_s \rangle \langle l=1, m_l | \vec{\varepsilon}^* \cdot \vec{r} | 3S \rangle$$

$$= \sum_{m_l} \langle 2P, J, J_z | m_l, m_s = +\tfrac{1}{2} \rangle \langle 2P, m_l | \vec{\varepsilon}^* \cdot \vec{r} | 3S \rangle$$

$$= \sum_{m_l} \langle 2P, J, J_z | m_l, m_s = +\tfrac{1}{2} \rangle \sqrt{\tfrac{1}{4\pi}} \left( \int r^3 dr \, R_{21}^* R_{30} \right) \vec{\varepsilon}^{*-k} K_k^{m_l}$$

$$= \sqrt{\tfrac{1}{4\pi}} R \sum_{m_l} \langle 2P, J, J_z | m_l, m_s = +\tfrac{1}{2} \rangle \vec{\varepsilon}^{*-k} K_k^{m_l}.$$

$$K_k^{m_l} = \begin{array}{c|ccc} & -1 & 0 & 1 \\ \hline -1 & -1 & & \\ 0 & & 1 & \\ 1 & & & -1 \end{array}$$

$$\frac{d\Gamma_{j^3=\frac{1}{2}}}{d\Omega} \propto \sum_{pol} \sum_{J, J_z} |\Delta H_{J J_z}|^2.$$

$$\sum_{J, J_z} |\Delta H_{J J_z}|^2 = \frac{1}{4\pi} R^2 \sum_{J, J_z} \sum_{m_l, m_l'} \langle m_l', m_s = +\tfrac{1}{2} | 2P, J, J_z \rangle \langle 2P, J, J_z | m_l, m_s = +\tfrac{1}{2} \rangle$$
$$(\vec{\varepsilon}^{*-k} K_k^{m_l}) (\vec{\varepsilon}^{-n} K_n^{m_l'})$$

$$= \frac{1}{4\pi} R^2 \sum_{m_l, m_l'} \langle m_l', m_s = +\tfrac{1}{2} | m_l, m_s = \tfrac{1}{2} \rangle (\vec{\varepsilon}^{*-k} K_k^{m_l})(\vec{\varepsilon}^{-n} K_n^{m_l'})$$

$$= \frac{1}{4\pi} R^2 \sum_{m_l, m_l'} \delta_{m_l m_l'} \vec{\varepsilon}^{*-k} K_k^{m_l} \vec{\varepsilon}^{-n} K_n^{m_l'}$$

$$= \frac{1}{4\pi} R^2 \sum_{m_l} K_k^{m_l} K_n^{m_l} \vec{\varepsilon}^{*-k} \vec{\varepsilon}^{-n}$$

$$= \frac{1}{4\pi} R^2 \; \delta_{kn} \; \vec{\varepsilon}^{\,*-k} \; \vec{\varepsilon}^{\,-n} \qquad ,$$

$$= \frac{1}{4\pi} R^2 \; \vec{\varepsilon} \cdot \vec{\varepsilon} \;, \qquad \vec{\varepsilon} \cdot \vec{\varepsilon} = 1 \; ! \; !$$

$$= \frac{1}{4\pi} R^2 \;, \text{ independent of angle.} \; //$$

(d) We need to calculate the amplitudes

$$a^1_{j_3} = \sum_{J_z} \langle 1S, j^3 = \pm\tfrac{1}{2} | \vec{\varepsilon}_2^{\,*} \cdot \vec{r} \,|\, 2P, J=\tfrac{1}{2}, J_z \rangle \langle 2P, J=\tfrac{1}{2}, J_z \,|\, \vec{\varepsilon}_1^{\,*} \cdot \vec{r} \,|\, 3S, j^3 = +\tfrac{1}{2} \rangle$$

and

$$a^2_{j_3} = \sum_{J_z} \langle 1S, j^3 = \pm\tfrac{1}{2} | \vec{\varepsilon}_2^{\,*} \cdot \vec{r} \,|\, 2P, J=\tfrac{3}{2}, J_z \rangle \langle 2P, J=\tfrac{3}{2}, J_z \,|\, \vec{\varepsilon}_1^{\,*} \cdot \vec{r} \,|\, 3S, j^3 = +\tfrac{1}{2} \rangle$$

I have suppressed factors that do not depend on angular quantum numbers.

$$\langle 1S, j^3 \,|\, \vec{\varepsilon}^{\,*} \cdot \vec{r} \,|\, 2P, J, J_z \rangle$$

$$= \sum_{m_\ell, m_s} \langle 1S, j^3 \,|\, \vec{\varepsilon}^{\,*} \cdot \vec{r} \,|\, \ell=1, m_\ell; s=\tfrac{1}{2}, m_s \rangle \langle \ell=1, m_\ell; s=\tfrac{1}{2}, m_s \,|\, 2P, J, J_z \rangle$$

$$= \sum_{m_\ell} \langle 1S \,|\, \vec{\varepsilon}^{\,*} \cdot \vec{r} \,|\, \ell=1, m_\ell \rangle \langle m_\ell, m_s = j^3 \,|\, 2P, J, J_z \rangle$$

$$= \sqrt{\tfrac{1}{4\pi}} \, R' \sum_{m_\ell} \vec{\varepsilon}^{\,*-k} K_k^{m_\ell} \langle m_\ell, m_s = j^3 \,|\, 2P, J, J_z \rangle$$

From 2.(c), $\langle 2P, J, J_z | \vec{\varepsilon}^{\,*} \cdot \vec{r} \,|\, 3S, j^3 = +\tfrac{1}{2} \rangle$

$$= \sqrt{\frac{1}{4\pi}}\, R \sum_{m_\ell} \langle 2P, J, J_z | m_\ell, m_s = \tfrac{1}{2} \rangle\, \varepsilon^{*-k} K_k^{m_\ell} .$$

Putting these together,

$$a'_{j_3} = \frac{1}{4\pi} R'R \sum_{m_\ell, m_\ell'} \sum_{J_z = -\frac{1}{2}}^{\frac{1}{2}} \langle m_\ell, m_s = j^3 | 2P, J=\tfrac{1}{2}, J_z \rangle \langle 2P, J=\tfrac{1}{2}, J_z | m_\ell', m_s = +\tfrac{1}{2} \rangle \times \vec{\varepsilon}_2^{\,*-k} K_k^{m_\ell}\, \vec{\varepsilon}_1^{\,*-k'} K_{k'}^{m_\ell'}$$

$$= \frac{1}{4\pi} R'R \sum_{m_\ell, m_\ell'} \Big\{ \langle m_\ell, m_s = j^3 | 2P, J=\tfrac{1}{2}, +\tfrac{1}{2} \rangle \langle 2P, J=\tfrac{1}{2}, +\tfrac{1}{2} | m_\ell', m_s = +\tfrac{1}{2} \rangle + \langle m_\ell, m_s = j^3 | 2P, J=\tfrac{1}{2}, -\tfrac{1}{2} \rangle \langle 2P, J=\tfrac{1}{2}, -\tfrac{1}{2} | m_\ell', m_s = +\tfrac{1}{2} \rangle \Big\} \times \vec{\varepsilon}_2^{\,*-k} K_k^{m_\ell}\, \vec{\varepsilon}_1^{\,*-k'} K_{k'}^{m_\ell'}$$

$$= \frac{1}{4\pi} R'R \sum_{m_\ell} \Big\{ \langle m_\ell, m_s = j^3 | 2P, J=\tfrac{1}{2}, +\tfrac{1}{2} \rangle \langle 2P, J=\tfrac{1}{2}, \tfrac{1}{2} | 0, +\tfrac{1}{2} \rangle\, \vec{\varepsilon}_1^{\,*-k'} K_{k'}^{0} + \langle m_\ell, m_s = j^3 | 2P, J=\tfrac{1}{2}, -\tfrac{1}{2} \rangle \langle 2P, J=\tfrac{1}{2}, -\tfrac{1}{2} | -1, +\tfrac{1}{2} \rangle\, \vec{\varepsilon}_1^{\,*-k'} K_{k'}^{-1} \Big\} \times \vec{\varepsilon}_2^{\,*-k} K_k^{m_\ell}$$

(see next page)

$$= \frac{1}{4\pi} R'R \sum_{m_\ell} \Big\{ -\sqrt{\tfrac{1}{3}} \langle m_\ell, m_s = j^3 | 2P, J=\tfrac{1}{2}, +\tfrac{1}{2} \rangle\, \cancel{\vec{\varepsilon}_1^{\,*0}} - \sqrt{\tfrac{2}{3}} \langle m_\ell, m_s = j^3 | 2P, J=\tfrac{1}{2}, -\tfrac{1}{2} \rangle\, \vec{\varepsilon}_1^{\,*+} \Big\}\, \vec{\varepsilon}_2^{\,*-k} K_k^{m_\ell}$$

$$\Rightarrow a'_{+\frac{1}{2}} = \frac{1}{4\pi} R'R \left(-\sqrt{\tfrac{2}{3}}\right) \Big\{ -\langle -1, +\tfrac{1}{2} | J=\tfrac{1}{2}, -\tfrac{1}{2} \rangle\, \vec{\varepsilon}_1^{\,*+}\, \vec{\varepsilon}_2^{\,*+} \Big\}$$

$$= \frac{1}{4\pi} R'R \left(\tfrac{2}{3}\right) \vec{\varepsilon}_1^{\,*+}\, \vec{\varepsilon}_2^{\,*+} \quad //$$

$$a^1_{-\frac{1}{2}} = \frac{1}{4\pi} R'R \left(-\sqrt{\frac{2}{3}}\right) \left\{ \langle 0, -\tfrac{1}{2} | J=\tfrac{1}{2}, -\tfrac{1}{2} \rangle \, \vec{\varepsilon}_1^{\,*+} \, \vec{\varepsilon}_2^{\,*0} \right\}$$

$$= \frac{1}{4\pi} R'R \left(-\frac{\sqrt{2}}{3}\right) \vec{\varepsilon}_1^{\,*+} \, \vec{\varepsilon}_2^{\,*0}$$

For $\varepsilon_1^{(1)} = (1, 0, 0)$, $\varepsilon_1^{(1)+} = -\frac{1}{\sqrt{2}}$, $\varepsilon_1^{(1)-} = \frac{1}{\sqrt{2}}$ $\varepsilon_1^{(1)0} = 0$

For $\varepsilon_1^{(2)} = (0, 1, 0)$, $\varepsilon_1^{(2)+} = -\frac{i}{\sqrt{2}}$, $\varepsilon_1^{(2)-} = -\frac{i}{\sqrt{2}}$ $\varepsilon_1^{(2)0} = 0$

· As before, For $\varepsilon_2^{(1)} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$,

$$\varepsilon_2^{(1)+} = -\frac{1}{\sqrt{2}}\cos\theta \, e^{i\phi}, \quad \varepsilon_2^{(1)-} = \frac{1}{\sqrt{2}}\cos\theta \, e^{-i\phi}, \quad \varepsilon_2^{(1)0} = -\sin\theta$$

For $\varepsilon_2^{(2)} = (\sin\phi, -\cos\phi, 0)$

$$\varepsilon_2^{(2)+} = \frac{i}{\sqrt{2}} e^{i\phi}, \quad \varepsilon_2^{(2)-} = \frac{i}{\sqrt{2}} e^{-i\phi}, \quad \varepsilon_2^{(2)0} = 0$$

$$\Rightarrow \sum_{pol} \sum_{j_3} |a^1_{j_3}|^2 = \left(\frac{R'R}{4\pi}\right)^2 \sum_{pol} \left\{ \frac{4}{9} \vec{\varepsilon}_1^{\,*+} \vec{\varepsilon}_1^{\,+} \, \vec{\varepsilon}_2^{\,*+} \varepsilon_2^{+} + \frac{2}{9} \vec{\varepsilon}_1^{\,*+} \vec{\varepsilon}_1^{\,+} \, \vec{\varepsilon}_2^{\,*0} \varepsilon_2^{0} \right\}$$

$$= \left(\frac{R'R}{4\pi}\right)^2 \left\{ \frac{4}{9}\left(\frac{1}{2}+\frac{1}{2}\right)\left(\frac{1}{2}\cos^2\theta + \frac{1}{2}\right) + \frac{2}{9}\left(\frac{1}{2}+\frac{1}{2}\right)\left(\sin^2\theta\right) \right\}$$

$$= \left(\frac{R'R}{4\pi}\right)^2 \left\{ \frac{2}{9} \right\} \quad // \quad \text{Isotropic!}$$

$$a^2_{j_3} = \frac{R'R}{4\pi} \sum_{m_\ell, m_\ell'} \sum_{J_z=-\frac{3}{2}}^{\frac{3}{2}} \langle m_\ell, m_j = j^3 | 2P, J=\tfrac{3}{2}, J_z \rangle \langle 2P, J=\tfrac{3}{2}, J_z | m_\ell', m_s = +\tfrac{1}{2} \rangle \; \vec{\varepsilon}_2^{\,*-k} K_k^{m_\ell} \; \vec{\varepsilon}_1^{\,*-k'} K_{k'}^{m_\ell'}$$

$$a^2_{+\frac{1}{2}} = \frac{RR'}{4\pi} \Big\{ \langle 1, \tfrac{1}{2} | J=\tfrac{3}{2}, \tfrac{3}{2} \rangle \langle J=\tfrac{3}{2}, \tfrac{3}{2} | 1, \tfrac{1}{2} \rangle \vec{\varepsilon}_2^{\,*-} \vec{\varepsilon}_1^{\,*-}$$

$$+ \langle 0\, \tfrac{1}{2} | J=\tfrac{3}{2}, \tfrac{1}{2} \rangle \langle J=\tfrac{3}{2}, \tfrac{1}{2} | 0, \tfrac{1}{2} \rangle \vec{\varepsilon}_2^{\,*0} \underbrace{\cancel{\vec{\varepsilon}_1^{\,*0}}}_{0}$$

$$+ \langle -1\, \tfrac{1}{2} | J=\tfrac{3}{2}, -\tfrac{1}{2} \rangle \langle J=\tfrac{3}{2}, -\tfrac{1}{2} | -1, \tfrac{1}{2} \rangle \; \vec{\varepsilon}_2^{\,*+} \vec{\varepsilon}_1^{\,*+} \Big\}$$

$$= \frac{RR'}{4\pi} \Big\{ \vec{\varepsilon}_2^{\,*-} \vec{\varepsilon}_1^{\,*-} + \tfrac{1}{3} \vec{\varepsilon}_2^{\,*+} \vec{\varepsilon}_1^{\,*+} \Big\}$$

$$a^2_{-\frac{1}{2}} = \frac{RR'}{4\pi} \Big\{ \langle 0, -\tfrac{1}{2} | J=\tfrac{3}{2}, -\tfrac{1}{2} \rangle \langle J=\tfrac{3}{2}, -\tfrac{1}{2} | -1, \tfrac{1}{2} \rangle \vec{\varepsilon}_2^{\,*0} \vec{\varepsilon}_1^{\,*+} (-1)$$

$$+ \langle 1, -\tfrac{1}{2} | J=\tfrac{3}{2}, +\tfrac{1}{2} \rangle \langle J=\tfrac{3}{2}, \tfrac{1}{2} | 0, \tfrac{1}{2} \rangle \vec{\varepsilon}_2^{\,*-} \underbrace{\cancel{\vec{\varepsilon}_1^{\,*0}}}_{0} (-1) \Big\}$$

$$= \frac{RR'}{4\pi} \Big\{ -\sqrt{\tfrac{2}{3}} \sqrt{\tfrac{1}{3}} \; \vec{\varepsilon}_2^{\,*0} \vec{\varepsilon}_1^{\,*+} \Big\}$$

$$\Rightarrow \sum_{pol} \sum_{j_3} | a^{(2)}_{j_3} |^2 = \left(\frac{RR'}{4\pi}\right)^2 \sum_{pol} \Big\{ \big| \vec{\varepsilon}_2^{\,*-} \vec{\varepsilon}_1^{\,*-} + \tfrac{1}{3} \vec{\varepsilon}_2^{\,*+} \vec{\varepsilon}_1^{\,*+} \big|^2 + \tfrac{2}{9} \big| \vec{\varepsilon}_2^{\,*0} \vec{\varepsilon}_1^{\,*+} \big|^2 \Big\}$$

$$= \left(\frac{RR'}{4\pi}\right)^2 \sum_{pol} \Big\{ \vec{\varepsilon}_2^{\,*-} \vec{\varepsilon}_2^{\,-} \vec{\varepsilon}_1^{\,*-} \vec{\varepsilon}_1^{\,-} + \tfrac{1}{3} \vec{\varepsilon}_2^{\,*-} \vec{\varepsilon}_2^{\,+} \vec{\varepsilon}_1^{\,*-} \vec{\varepsilon}_1^{\,+} + \tfrac{1}{3} \vec{\varepsilon}_2^{\,-} \vec{\varepsilon}_2^{\,*+} \vec{\varepsilon}_1^{\,-} \vec{\varepsilon}_1^{\,*+}$$

$$+ \tfrac{1}{9} \vec{\varepsilon}_2^{\,*+} \vec{\varepsilon}_2^{\,+} \vec{\varepsilon}_1^{\,*+} \vec{\varepsilon}_1^{\,+} + \tfrac{2}{9} \vec{\varepsilon}_2^{\,*0} \vec{\varepsilon}_2^{\,0} \vec{\varepsilon}_1^{\,*+} \vec{\varepsilon}_1^{\,+} \Big\}$$

$$= \left(\frac{RR'}{4\pi}\right)^2 \left\{ \left(\frac{1}{2}\cos^2\theta + \frac{1}{2}\right)\left(\frac{1}{2}+\frac{1}{2}\right) + \frac{1}{3}\left(-\frac{1}{2}\cos^2\theta\, e^{2i\phi} + \frac{1}{2}e^{2i\phi}\right)\cancelto{0}{\left(-\frac{1}{2}+\frac{1}{2}\right)} \right.$$

$$+ \frac{1}{3}\left(-\frac{1}{2}\cos^2\theta\, e^{-2i\phi} + \frac{1}{2}e^{-2i\phi}\right)\cancelto{0}{\left(-\frac{1}{2}+\frac{1}{2}\right)}$$

$$\left. + \frac{1}{9}\left(\frac{1}{2}\cos^2\theta + \frac{1}{2}\right)\left(\frac{1}{2}+\frac{1}{2}\right) + \frac{2}{9}\sin^2\theta\left(\frac{1}{2}+\frac{1}{2}\right) \right\}$$

$$= \left(\frac{RR'}{4\pi}\right)^2 \left\{ \frac{1}{2}\cos^2\theta + \frac{1}{2} + \frac{1}{18}\cos^2\theta + \frac{1}{18} + \frac{2}{9}\sin^2\theta \right\}$$

$$= \left(\frac{RR'}{4\pi}\right)^2 \left(\frac{1}{3}\cos^2\theta + \frac{7}{9}\right) \quad //$$

This contribution is angle dependent.