Solution Set 5

1. (a) $\psi(t) = \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} = \alpha(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$H_0 = -\frac{E_0}{2}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$\psi(t) = e^{-iH_0 t}\,\psi(0)$$

$$e^{-iH_0 t} = \exp\left\{ \frac{iE_0}{2} t \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \right\}$$

$$= \begin{pmatrix} e^{i\frac{E_0}{2}t} & \\ & e^{-i\frac{E_0}{2}t} \end{pmatrix}$$

$$\Rightarrow \psi(t) = \begin{pmatrix} e^{i\frac{E_0}{2}t} & \\ & e^{-i\frac{E_0}{2}t} \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\frac{E_0}{2}t} \\ e^{-i\frac{E_0}{2}t} \end{pmatrix}$$

$$\Rightarrow \alpha(t) = \frac{1}{\sqrt{2}} e^{i\frac{E_0}{2}t}, \quad \beta(t) = \frac{1}{\sqrt{2}} e^{-i\frac{E_0}{2}t}$$

For a spin in a magnetic field, the hamiltonian is $H = -\vec{\mu}_s \cdot \vec{B}$, where $\vec{\mu}_s = g\frac{q}{2m}\vec{S}$ is the spin magnetic moment.

Assume the magnetic field is constant and directed along the z-axis, then

$$H = -g\frac{q}{2m} S_z B_0$$

$$= -\frac{gqB_0}{4m}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$= H_0 \quad \text{when we identify} \quad \frac{gqB}{2m} = E_0 .$$

When the spin starts out in the state $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$, which is an eigenstate of S_x, we expect to see precession since the magnetic field is not aligned with the spin. ψ will return to its current state after a period T given by

$$\frac{E_0}{2}T = 2n\pi \quad \Rightarrow T = \frac{4\pi}{E_0} \quad , \text{ i.e. } \omega_0 = \frac{E_0}{4\pi}$$

Also, ψ oscillates between S_x eigen state & S_y eigenstate, while S_z remains the same.

$$\langle S_x(t)\rangle = \langle \psi(t)|\, S_x \,|\psi(t)\rangle = \frac{1}{2}\left(e^{-i\frac{E_0}{2}t} \quad e^{i\frac{E_0}{2}t}\right)\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{i\frac{E_0}{2}t} \\ e^{-i\frac{E_0}{2}t} \end{pmatrix}$$

$$= \frac{1}{4}\left(e^{iE_0t} + e^{-iE_0t}\right) = \frac{1}{2}\cos(E_0t)$$

$$\langle S_y(t)\rangle = \frac{1}{4}\left(e^{-i\frac{E_0}{2}t} \quad e^{i\frac{E_0}{2}t}\right)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} e^{i\frac{E_0}{2}t} \\ e^{-i\frac{E_0}{2}t} \end{pmatrix}$$

$$= \frac{1}{4}\left(ie^{iE_0t} - ie^{-iE_0t}\right) = -\frac{1}{2}\sin(E_0t)$$

$$\langle s_z(t)\rangle = \frac{1}{4}\begin{pmatrix} e^{-i\frac{E_0}{2}t} & e^{i\frac{E_0}{2}t}\end{pmatrix}\begin{pmatrix}1 & \\ & -1\end{pmatrix}\begin{pmatrix} e^{iE_0/2\,t} \\ e^{-iE_0/2\,t}\end{pmatrix}$$

$$= \frac{1}{4}(1-1)$$

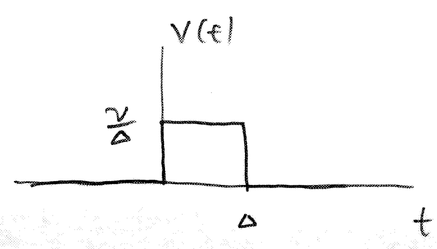
$$= 0 \,//$$

(b) We'll first solve for $\psi_I(t)$ in the interaction picture, and find the shrodinger picture wave function $\psi_s(t)$ using the relation

$$|\psi_I(t)\rangle = e^{iH_0 t}\,|\psi_s(t)\rangle .$$

Formally, $|\psi_I(t)\rangle = T\exp\left\{-i\int_0^t V_I(t)\,dt\right\}|\psi_I(0)\rangle$

$$V(t) = \begin{cases} \begin{pmatrix} 0 & \nu/\Delta \\ \nu/\Delta & 0\end{pmatrix}, & 0 \le t \le \Delta \\ 0 & \text{otherwise}\end{cases}$$



Using the step function $V(t) = \frac{\nu}{\Delta}\big(\theta(t) - \theta(t-\Delta)\big)\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$

$$V_I(t) = e^{iH_0 t}\,V(t)\,e^{-iH_0 t}$$

$$= \frac{\nu}{\Delta}\big(\theta(t)-\theta(t-\Delta)\big)\begin{pmatrix} e^{-i\frac{E_0}{2}t} & \\ & e^{i\frac{E_0}{2}t}\end{pmatrix}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} e^{i\frac{E_0}{2}t} & \\ & e^{-i\frac{E_0}{2}t}\end{pmatrix}$$

$$= \frac{\nu}{\Delta}\left(\theta(t) - \theta(t-\Delta)\right)\begin{pmatrix} 0 & e^{-iE_0 t} \\ e^{iE_0 t} & 0 \end{pmatrix}$$

$$\int_0^t V_I(t')\,dt' = \frac{\nu}{\Delta}\int_0^\Delta dt' \begin{pmatrix} 0 & e^{-iE_0 t} \\ e^{iE_0 t} & 0 \end{pmatrix}$$

$$= \frac{\nu}{\Delta}\begin{pmatrix} 0 & \frac{i}{E_0}\left(e^{-iE_0\Delta} - 1\right) \\ \frac{-i}{E_0}\left(e^{iE_0\Delta} - 1\right) & 0 \end{pmatrix}$$

$$\simeq \frac{\nu}{\Delta E_0}\begin{pmatrix} 0 & E_0\Delta \\ E_0\Delta & 0 \end{pmatrix} \quad \text{to } 1^{st} \text{ order in } E_0\Delta$$

$$= \nu\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{independent of } \Delta.$$

$$\exp\left\{-i\int V_I(t')\,dt'\right\} = \exp\left\{-i\nu\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}$$

$$= I - i\nu\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{(-i\nu)^2}{2!}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{(-i\nu)^3}{3!}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{(-i\nu)^4}{4!}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 1 - \frac{\nu^2}{2!} + \frac{\nu^4}{4!} + \cdots & -i\nu + \frac{i\nu^3}{3!} - \cdots \\ -i\nu + \frac{i\nu^3}{3!} + \cdots & 1 - \frac{\nu^2}{2!} + \frac{\nu^4}{4!} + \cdots \end{pmatrix}$$

$$= \begin{pmatrix} \cos\nu & -i\sin\nu \\ -i\sin\nu & \cos\nu \end{pmatrix}$$

$$\therefore \quad \psi_I(t) = \begin{pmatrix} \cos\nu & -i\sin\nu \\ -i\sin\nu & \cos\nu \end{pmatrix} \psi_I(0)$$

$$\Rightarrow \psi_s(t) = e^{-iH_0 t} \begin{pmatrix} \cos\nu & -i\sin\nu \\ -i\sin\nu & \cos\nu \end{pmatrix} \psi_s(0)$$

$$= \begin{pmatrix} e^{i\frac{E_0}{2}t} & \\ & e^{-i\frac{E_0}{2}t} \end{pmatrix} \begin{pmatrix} \cos\nu & -i\sin\nu \\ -i\sin\nu & \cos\nu \end{pmatrix} \psi_s(0)$$

This has the nice interpretation that at $t=0$ the perturbing hamiltonian acts on $\psi_s(0)$, then the wave function evolves freely due to the unperturbed hamiltonian.

For $\psi(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we get

$$\psi(t) = \begin{pmatrix} e^{i\frac{E_0}{2}t} \cos\nu \\ -i e^{-i\frac{E_0}{2}t} \sin\nu \end{pmatrix}$$

c. since $V^2 = \frac{\nu^2}{\Delta^2} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \times \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \frac{\nu^2}{\Delta^2} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$, a good guess for the perturbing δ- function hamiltonian is

$$V' = \begin{pmatrix} 0 & \nu/\Delta \\ \nu/\Delta & 0 \end{pmatrix}.$$

Then, $\psi(T) = \begin{pmatrix} \cos\nu' & -i\sin\nu' \\ -i\sin\nu' & \cos\nu' \end{pmatrix} \underbrace{\begin{pmatrix} e^{i\frac{E_0}{2}T} & \\ & e^{-i\frac{E_0}{2}T} \end{pmatrix} \begin{pmatrix} \cos\nu & -i\sin\nu \\ -i\sin\nu & \cos\nu \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{= \begin{pmatrix} e^{i\frac{E_0}{2}T}\cos\nu \\ -ie^{-i\frac{E_0}{2}T}\sin\nu \end{pmatrix}}$

$$\Rightarrow \psi(T) = \begin{pmatrix} e^{i\frac{E_0}{2}T}\cos\nu\cos\nu' - e^{-i\frac{E_0}{2}T}\sin\nu\sin\nu' \\ -ie^{i\frac{E_0}{2}T}\cos\nu\sin\nu' - ie^{-i\frac{E_0}{2}T}\sin\nu\cos\nu' \end{pmatrix}$$

$$= e^{i\frac{E_0}{2}T} \begin{pmatrix} \cos\nu\cos\nu' - e^{-iE_0T}\sin\nu\sin\nu' \\ -i\cos\nu\sin\nu' - ie^{-iE_0T}\sin\nu\cos\nu' \end{pmatrix}$$

We want this to be $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ up to a phase.

For $T = \frac{2\pi}{E_0}$,

$$\psi(T) = e^{i\pi} \begin{pmatrix} \cos\nu\cos\nu' - \sin\nu\sin\nu' \\ -i(\cos\nu\sin\nu' + \sin\nu\cos\nu') \end{pmatrix}$$

$$= e^{i\pi} \begin{pmatrix} \cos(\nu+\nu') \\ -i\sin(\nu+\nu') \end{pmatrix}$$

$\Rightarrow$ $\underline{\underline{\nu' = -\nu}}$ will give the desired result.

For $T = \frac{\pi}{E_0}$,

$$\psi(T) = e^{i\frac{\pi}{2}} \begin{pmatrix} \cos\nu\cos\nu' + \sin\nu\sin\nu' \\ -i\cos\nu\sin\nu' + i\sin\nu\cos\nu' \end{pmatrix}$$

$$= e^{i\frac{\pi}{2}} \begin{pmatrix} \cos(\nu-\nu') \\ -i\sin(\nu-\nu') \end{pmatrix}$$

$$\Rightarrow \underline{\underline{\nu' = \nu}}$$

For $T = \frac{\pi}{2E_0}$,

$$\psi(T) = e^{i\frac{\pi}{4}} \begin{pmatrix} \cos\nu\cos\nu' + i\sin\nu\sin\nu' \\ -i\cos\nu\sin\nu' - i\sin\nu\cos\nu' \end{pmatrix}$$

$$\cos\nu\cos\nu' + i\sin\nu\sin\nu' = 1 \qquad (1)$$

$$i\cos\nu\sin\nu' + \sin\nu\cos\nu' = 0 \qquad (2)$$

$$(2) \Rightarrow \quad i\tan\nu' + \tan\nu = 0$$

$$\Rightarrow \nu' = \arctan(i\tan\nu) \quad //$$

plugging this into (1),

$$\cos\nu + i\sin\nu\tan\nu' = \sec\nu'$$

$$\cos\nu + i\sin\nu(i\tan\nu) = \sqrt{1+(i\tan\nu)^2}$$

$$\Rightarrow \cos\nu - \frac{\sin^2\nu}{\cos\nu} = \sqrt{1-\tan^2\nu}$$

$$1 - \tan^2\nu = 1 - \tan^2\nu \quad \checkmark$$

For a general T,

We want $0 = -i\cos\nu\sin\nu' - i e^{-iE_0T}\sin\nu\cos\nu'$

$$\Rightarrow \quad 0 = \cos\nu\sin\nu' + e^{-iE_0T}\sin\nu\cos\nu'$$

$$\sin\nu\cos\nu' e^{-iE_0T} = -\cos\nu\sin\nu'$$

$$\tan\nu\cot\nu' = -e^{iE_0T}$$

$$\cot\nu' = -e^{iE_0T}\cot\nu$$

$$\nu' = -\operatorname{Arccot}\left(e^{iE_0T}\cot\nu\right)$$

2. (a)

$$H_r = \begin{pmatrix} E_a & \frac{V}{2}e^{i\omega t} \\ \frac{V}{2}e^{-i\omega t} & E_b \end{pmatrix}$$

$$H_r^+ = \begin{pmatrix} E_a & \frac{V}{2}e^{-i\omega t} \\ \frac{V}{2}e^{i\omega t} & E_b \end{pmatrix}^T = \begin{pmatrix} E_a & \frac{V}{2}e^{i\omega t} \\ \frac{V}{2}e^{-i\omega t} & E_b \end{pmatrix} = H_r$$

$\Rightarrow$ H_r is hermitian.

This means the time evolution operator, given by

$U(t, t') = T\exp\left\{-i\int_{t'}^{t} H_r(t'')\,dt''\right\}$ is unitary, so it's effect is to rotate ψ in the 2-complex dimensional, space. The norm of ψ at any time remains const.

$\Rightarrow$ $|\alpha(t)|^2 + |\beta(t)|^2 = 1$ after normalizing ψ.

(b) We'll solve this in two ways. The first involves solving 2^{nd} order ODE. The second method (I learned it from Michael Reskin) is simpler and more elegant, and we solve a very simple 1^{st} order ODE.

Way 1

Let $\Psi(t) = C_a(t)\, e^{-iE_a t}\begin{pmatrix}1\\0\end{pmatrix} + C_b(t)\, e^{-iE_b t}\begin{pmatrix}0\\1\end{pmatrix}$

Then, Schrod. eqn. leads to the following system of equations for C_a & C_b (see Griffiths 9.1):

$$\dot{C}_a = -i V_{ab}\, e^{-i(E_b - E_a)t}\, C_b \qquad (1)$$

$$\dot{C}_b = -i V_{ba}\, e^{i(E_b - E_a)t}\, C_a , \qquad (2)$$

where $V_{ab} = \langle \psi_a | V | \psi_b \rangle$, $V = \frac{V}{2}\begin{pmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & 0 \end{pmatrix}$

$$V_{ab} = (1 \quad 0)\, \frac{V}{2}\begin{pmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & 0 \end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$$

$$= \frac{V}{2}\, e^{i\omega t}$$

$$V_{ba} = (0 \quad 1)\, \frac{V}{2}\begin{pmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & 0 \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$= \frac{V}{2}\, e^{-i\omega t}$$

(1) →
$$C_b = i\, (V_{ab})^{-1}\, e^{i(E_b - E_a)t}\, \dot{C}_a$$

$$C_b = i\, \frac{2}{V}\, e^{i(E_b - E_a - \omega)t}\, \dot{C}_a \qquad (3)$$

✓

$$\frac{d}{dt}(3) \Rightarrow \quad \dot{C}_b = i\frac{2}{V}\left\{ i(E_b - E_a - \omega)\dot{C}_a + \ddot{C}_a \right\} e^{i(E_b - E_a - \omega)t} \qquad (4)$$

$(2) \to (4) \Rightarrow$

$$-i\frac{V}{2} \cancel{e^{i(E_b - E_a - \omega)t}} C_a = \frac{i2}{V}\left(i(E_b - E_a - \omega)\dot{C}_a + \ddot{C}_a \right) \cancel{e^{i(E_b - E_a - \omega)t}}$$

$$-\frac{V^2}{4} C_a = i(E_b - E_a - \omega)\dot{C}_a + \ddot{C}_a$$

$$\ddot{C}_a + i\lambda \dot{C}_a + \frac{V^2}{4} C_a = 0 \quad , \quad \lambda = E_b - E_a - \omega \qquad \checkmark \qquad (5)$$

Laplace transforming,

$$\int_0^\infty e^{-st}\ddot{C}_a\, dt + i\lambda \int_0^\infty e^{-st}\dot{C}_a\, dt + \frac{V^2}{4}\int_0^\infty e^{-st} C_a = 0 \qquad (6)$$

$$\int_0^\infty e^{-st}\ddot{C}_a = \int_0^\infty \frac{d}{dt}\left(e^{-st}\dot{C}_a\right) - \int_0^\infty (-s) e^{-st}\dot{C}_a$$

$$= e^{-st}\dot{C}_a \Big|_0^\infty + s\int_0^\infty e^{-st}\dot{C}_a\, dt$$

$$= -\dot{C}_a(0) + s\int_0^\infty \frac{d}{dt}\left(e^{-st} C_a\right) dt - s\int_0^\infty (-s) e^{-st} C_a\, dt$$

$$= -\dot{C}_a(0) - s\,C_a(0) + s^2 \int_0^\infty e^{-st} C_a$$

$$(6) \to -\dot{C}_a(0) - s\,C_a(0) + s^2\int_0^\infty e^{-st} C_a(t) - i\lambda C_a(0) + i\lambda s\int_0^\infty e^{-st} C_a + \frac{V^2}{4}\int_0^\infty e^{-st} C_a$$

$$= 0$$

$$\Rightarrow \left(s^2 + i\lambda s + \frac{V^2}{4}\right)\int_0^\infty e^{-st} C_a(t) = \dot{C}_a(0) + (s + i\lambda)\, C_a(0) \qquad \checkmark$$

$$L[C_a](s) = \frac{\dot{C}_a(0) + (s+id)\,C_a(0)}{s^2 + ids + \frac{V^2}{4}}$$

$$= \frac{\dot{C}_a(0) + (s+id)\,C_a(0)}{s^2 + 2i\frac{d}{2}s + \left(\frac{id}{2}\right)^2 - \left(\frac{id}{2}\right)^2 + \frac{V^2}{4}}$$

$$= \frac{\dot{C}_a(0) + (s+id)\,C_a(0)}{\left(s+\frac{id}{2}\right)^2 + \frac{1}{4}(d^2+V^2)}$$

Since $\psi(t=0) = \begin{pmatrix}1\\0\end{pmatrix}$, $C_a(0) = 1$, $C_b(0) = 0$

$$\frac{d\psi}{dt} = (\dot{C}_a - iE_aC_a)e^{-iE_at}\begin{pmatrix}1\\0\end{pmatrix} + (\dot{C}_b - iE_bC_b)\,e^{-iE_bt}\begin{pmatrix}0\\1\end{pmatrix}$$

$$\left.\frac{d\psi}{dt}\right|_{t=0} = (\dot{C}_a(0) - iE_a)\begin{pmatrix}1\\0\end{pmatrix} + \dot{C}_b\begin{pmatrix}0\\1\end{pmatrix}$$

$$= -iH(t=0)\,\psi(t=0) = -i\begin{pmatrix}E_a & \frac{V}{2}\\ \frac{V}{2} & E_b\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = -iE_a\begin{pmatrix}1\\0\end{pmatrix} - i\frac{V}{2}\begin{pmatrix}0\\1\end{pmatrix}$$

$$\Rightarrow \dot{C}_a(0) = 0, \quad \dot{C}_b(0) = -i\frac{V}{2}$$

$$\Rightarrow L[C_a](s) = \frac{s+id}{\left(s+\frac{id}{2}\right)^2 + \frac{1}{4}(d^2+V^2)}$$ ✓

Two poles at $s + \frac{id}{2} = \pm\frac{i}{2}\sqrt{d^2+V^2} \Rightarrow s = \frac{i}{2}\left(-d \pm \sqrt{d^2+V^2}\right)$

Inverting the Laplace transform,

$$C_a(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\left\{\frac{s+i\lambda}{\left(s+\frac{i\lambda}{2}\right)^2+\frac{1}{4}(\lambda^2+V^2)}\right\} ds$$

$$= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}(s+i\lambda)\frac{1}{\left(s+\frac{i}{2}\left(\lambda+\sqrt{\lambda^2+V^2}\right)\right)\left(s+\frac{i}{2}\left(\lambda-\sqrt{\lambda^2+V^2}\right)\right)}$$

$$= \left\{ \exp\left\{-\frac{i}{2}\left(\lambda+\sqrt{\lambda^2+V^2}\right)t\right\}\left(\frac{\frac{i\lambda}{2}-\frac{i}{2}\sqrt{\lambda^2+V^2}}{\left(-\frac{i}{2}\lambda-\frac{i}{2}\sqrt{\lambda^2+V^2}+\frac{i\lambda}{2}-\frac{i}{2}\sqrt{\lambda^2+V^2}\right)}\right)\right.$$

$$\left. + \exp\left\{\frac{i}{2}\left(-\lambda+\sqrt{\lambda^2+V^2}\right)t\right\}\left(\frac{\frac{i\lambda}{2}+\frac{i}{2}\sqrt{\lambda^2+V^2}}{\left(-\frac{i}{2}\lambda+\frac{i}{2}\sqrt{\lambda^2+V^2}+\frac{i}{2}\lambda+\frac{i}{2}\sqrt{\lambda^2+V^2}\right)}\right)\right\}$$

$$= \left\{ \exp\left\{\left(-\frac{i\lambda}{2}-\frac{i}{2}\sqrt{\lambda^2+V^2}\right)t\right\}\left(\frac{\frac{i\lambda}{2}-\frac{i}{2}\sqrt{\lambda^2+V^2}}{-i\sqrt{\lambda^2+V^2}}\right)\right.$$

$$\left. + \exp\left\{\left(-\frac{i\lambda}{2}+\frac{i}{2}\sqrt{\lambda^2+V^2}\right)t\right\}\left(\frac{\frac{i\lambda}{2}+\frac{i}{2}\sqrt{\lambda^2+V^2}}{i\sqrt{\lambda^2+V^2}}\right)\right.$$

$$= \frac{1}{\sqrt{\lambda^2+V^2}}\left\{ \exp\left\{\left(-\frac{i\lambda}{2}-\frac{i}{2}\sqrt{\lambda^2+V^2}\right)t\right\}\left(-\frac{\lambda}{2}+\frac{1}{2}\sqrt{\lambda^2+V^2}\right)\right.$$

$$\left. + \exp\left\{\left(-\frac{i\lambda}{2}+\frac{i}{2}\sqrt{\lambda^2+V^2}\right)t\right\}\left(\frac{\lambda}{2}+\frac{1}{2}\sqrt{\lambda^2+V^2}\right)\right\}$$

$$= \frac{1}{\sqrt{\lambda^2+V^2}} e^{-i\frac{\lambda}{2}t} \left\{ \frac{\lambda}{2} \left(e^{\frac{i}{2}\sqrt{\lambda^2+V^2}\,t} - e^{-\frac{i}{2}\sqrt{\lambda^2+V^2}\,t} \right) \right.$$

$$\left. + \frac{\sqrt{\lambda^2+V^2}}{2} \left(e^{\frac{i}{2}\sqrt{\lambda^2+V^2}\,t} + e^{-\frac{i}{2}\sqrt{\lambda^2+V^2}\,t} \right) \right\}$$

$$= e^{-i\frac{\lambda}{2}t} \left\{ \frac{i\lambda}{\sqrt{\lambda^2+V^2}} \sin\left(\tfrac{1}{2}\sqrt{\lambda^2+V^2}\,t\right) + \cos\left(\tfrac{1}{2}\sqrt{\lambda^2+V^2}\,t\right) \right\}$$

So,

$$\boxed{C_a(t) = e^{-\frac{i}{2}(E_b-E_a-\omega)t} \left\{ i\frac{(E_b-E_a-\omega)}{\sqrt{(E_b-E_a-\omega)^2+V^2}} \sin\left(\tfrac{1}{2}\sqrt{(E_b-E_a-\omega)^2+V^2}\,t\right) + \cos\left(\tfrac{1}{2}\sqrt{(E_b-E_a-\omega)^2+V^2}\,t\right) \right\}}$$

$$C_b(t) = i\frac{2}{V} e^{i\lambda t}\, \dot{C}_a(t)$$

$$\dot{C}_a(t) = -i\frac{\lambda}{2} e^{-i\frac{\lambda}{2}t} \left\{ \frac{i\lambda}{\sqrt{\lambda^2+V^2}} \sin\left(\tfrac{1}{2}\sqrt{\lambda^2+V^2}\,t\right) + \cos\left(\tfrac{1}{2}\sqrt{\lambda^2+V^2}\,t\right) \right\}$$

$$+ e^{-i\frac{\lambda}{2}t} \left\{ \frac{i\lambda}{\sqrt{\lambda^2+V^2}} \tfrac{1}{2}\sqrt{\lambda^2+V^2} \cos\left(\tfrac{1}{2}\sqrt{\lambda^2+V^2}\,t\right) + \tfrac{1}{2}\sqrt{\lambda^2+V^2} \sin\left(\tfrac{1}{2}\sqrt{\lambda^2+V^2}\,t\right) \right\}$$

$$= e^{-i\frac{\lambda}{2}t} \left\{ \left(\frac{\lambda^2}{2\sqrt{\lambda^2+V^2}} - \frac{\lambda^2+V^2}{2\sqrt{\lambda^2+V^2}} \right) \sin\left(\tfrac{1}{2}\sqrt{\lambda^2+V^2}\,t\right) \right.$$

$$\left. \left(-\frac{i\lambda}{2} \cancel{+} \frac{i\lambda}{2} \right) \cos\left(\tfrac{1}{2}\sqrt{\lambda^2+V^2}\,t\right) \right\}$$

$$= \frac{-V^2}{2\sqrt{\lambda^2+V^2}} e^{-i\frac{\lambda}{2}t} \sin\left(\tfrac{1}{2}\sqrt{\lambda^2+V^2}\,t\right)$$

$$\Rightarrow \boxed{C_b(t) = \frac{-iV}{\sqrt{\Lambda^2+V^2}}\, e^{i\frac{\Lambda}{2}t} \sin\left(\tfrac{1}{2}\sqrt{\Lambda^2+V^2}\, t\right)}$$

$C_b(0) = 0$, ✓

$$\dot{c}_b = \frac{-V}{\sqrt{\Lambda^2+V^2}}\, e^{i\frac{\Lambda}{2}t}\left[\frac{i\Lambda}{2}\sin\left(\tfrac{1}{2}\sqrt{\Lambda^2+V^2}\, t\right) + \tfrac{1}{2}\sqrt{\Lambda^2+V^2}\cos\left(\tfrac{1}{2}\sqrt{\Lambda^2+V^2}\, t\right)\right]$$

$\dot{c}_b(0) = -i\frac{V}{2}$ ✓

<u>Way 2</u> (simpler!)

$$i \frac{\partial}{\partial t} \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} = \begin{pmatrix} E_a & \frac{V}{2} e^{i\omega t} \\ \frac{V}{2} e^{-i\omega t} & E_b \end{pmatrix} \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix}$$

$$\Rightarrow \quad i\dot{\alpha} = E_a \alpha + \frac{V}{2} e^{i\omega t} \beta \qquad (1)$$

$$i\dot{\beta} = E_b \beta + \frac{V}{2} e^{-i\omega t} \alpha \qquad (2)$$

Ansatz:
$$\alpha(t) = \tilde{\alpha}(t) e^{i\frac{\omega}{2}t} e^{-i\left(\frac{E_a+E_b}{2}\right)t}$$

$$\beta(t) = \tilde{\beta}(t) e^{-i\frac{\omega}{2}t} e^{-i\left(\frac{E_a+E_b}{2}\right)t}$$

$$(1) \Rightarrow i\left(\dot{\tilde{\alpha}} + \frac{i\omega}{2}\tilde{\alpha} - i\left(\frac{E_a+E_b}{2}\right)\tilde{\alpha}\right) e^{i\frac{\omega}{2}t} e^{-i\left(\frac{E_a+E_b}{2}\right)t}$$

$$= E_a \tilde{\alpha} e^{i\frac{\omega}{2}t} e^{-i\left(\frac{E_a+E_b}{2}\right)t} + \frac{V}{2} e^{i\frac{\omega}{2}t} \tilde{\beta} e^{-i\left(\frac{E_a+E_b}{2}\right)t}$$

$$\Rightarrow \quad \dot{\tilde{\alpha}} = -\frac{i\omega}{2}\tilde{\alpha} + i\left(\frac{E_a+E_b}{2}\right)\tilde{\alpha} - iE_a\tilde{\alpha} - i\frac{V}{2}\tilde{\beta}$$

$$\dot{\tilde{\alpha}} = \frac{i}{2}(E_b - E_a - \omega)\tilde{\alpha} - \frac{i}{2}V\tilde{\beta} \qquad (3)$$

$$(2) \Rightarrow i\left(\dot{\tilde{\beta}} - \frac{i\omega}{2}\tilde{\beta} - i\left(\frac{E_a+E_b}{2}\right)\tilde{\beta}\right) e^{-i\frac{\omega}{2}t} e^{-i\left(\frac{E_a+E_b}{2}\right)t}$$

$$= E_b \tilde{\beta} e^{-i\frac{\omega}{2}t} e^{-i\left(\frac{E_a+E_b}{2}\right)t} + \frac{V}{2} e^{-i\frac{\omega}{2}t} \tilde{\alpha} e^{-i\left(\frac{E_a+E_b}{2}\right)t}$$

$$\Rightarrow \quad \dot{\tilde{\beta}} = \frac{i\omega}{2}\tilde{\beta} + i\left(\frac{E_a+E_b}{2}\right)\tilde{\beta} - iE_b\tilde{\beta} - i\frac{V}{2}\tilde{\alpha}$$

$$\dot{\tilde{\beta}} = -\frac{i}{2}(E_b - E_a - \omega)\tilde{\beta} - i\frac{V}{2}\tilde{\alpha} \qquad (4)$$

(3) & (4) $\Rightarrow$ $$i\frac{d}{dt}\begin{pmatrix}\tilde{\alpha}\\ \tilde{\beta}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-(E_b-E_a-\omega) & V\\ V & E_b-E_a-\omega\end{pmatrix}\begin{pmatrix}\tilde{\alpha}\\ \tilde{\beta}\end{pmatrix}$$

The matrix on the right hand has no time dependence, so we can solve this equation by diagonalizing the hamiltonian $$H' = \frac{1}{2}\begin{pmatrix}-(E_b-E_a-\omega) & V\\ V & E_b-E_a-\omega\end{pmatrix}$$

Let $\lambda = E_b-E_a-\omega$, $\omega_r = \frac{1}{2}\sqrt{\lambda^2+V^2}$;

$H' = \frac{1}{2}\begin{pmatrix}-\lambda & V\\ V & \lambda\end{pmatrix}$, then, the eigen vectors & values of H' are

$$H'\begin{pmatrix}-\frac{\lambda+2\omega_r}{V}\\ 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\frac{\lambda^2+2\omega_r\lambda+V^2}{V}\\ -\lambda-2\omega_r+\lambda\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\frac{4\omega_r^2+2\omega_r\lambda}{V}\\ -2\omega_r\end{pmatrix}$$

$$= -\omega_r\begin{pmatrix}-\frac{\lambda+2\omega_r}{V}\\ 1\end{pmatrix}$$

$$H'\begin{pmatrix}-\frac{\lambda-2\omega_r}{V}\\ 1\end{pmatrix} = \begin{pmatrix}\frac{\lambda^2-2\omega_r\lambda+V^2}{V}\\ -\lambda+2\omega_r+\lambda\end{pmatrix} = \begin{pmatrix}\frac{4\omega_r^2-\lambda\omega_r}{V}\\ \omega_r\end{pmatrix}$$

$$= \omega_r\begin{pmatrix}-\frac{\lambda-2\omega_r}{V}\\ 1\end{pmatrix}$$

So $$H' = \begin{pmatrix}-\frac{\lambda+2\omega_r}{V} & -\frac{\lambda-2\omega_r}{V}\\ 1 & 1\end{pmatrix}\begin{pmatrix}-\omega_r & \\ & \omega_r\end{pmatrix}\begin{pmatrix}-\frac{\lambda+2\omega_r}{V} & -\frac{\lambda-2\omega_r}{V}\\ 1 & 1\end{pmatrix}^{-1}$$

$$H' = A \begin{pmatrix} -\omega_r & \\ & \omega_r \end{pmatrix} A^{-1}$$

Then, $\begin{pmatrix} \tilde{\alpha}(t) \\ \tilde{\beta}(t) \end{pmatrix} = \exp\{-iH't\} \begin{pmatrix} \tilde{\alpha}(0) \\ \tilde{\beta}(0) \end{pmatrix}$,

$$\exp\{-iH't\} = \exp\left\{-iA \begin{pmatrix} -\omega_r & \\ & \omega_r \end{pmatrix} A^{-1}\right\}$$

$$= 1 - iA\begin{pmatrix} -\omega_r & \\ & \omega_r \end{pmatrix}A^{-1} + \frac{1}{2!}(-i)^2 A\begin{pmatrix} -\omega_r & \\ & \omega_r \end{pmatrix}A^{-1}A\begin{pmatrix} -\omega_r & \\ & \omega_r \end{pmatrix}A^{-1}$$

$$+ \cdots$$

$$= A\left\{ I - i\begin{pmatrix} -\omega_r & \\ & \omega_r \end{pmatrix} + \frac{(-i)^2}{2!}\begin{pmatrix} -\omega_r & \\ & \omega_r \end{pmatrix}^2 + \cdots \right\} A^{-1}$$

$$= A \begin{pmatrix} e^{i\omega_r} & \\ & e^{-i\omega_r} \end{pmatrix} A^{-1}$$

$$= \begin{pmatrix} \cos\omega_r + \frac{i\Delta}{2\omega_r}\sin\omega_r & -\frac{iV}{2\omega_r}\sin\omega_r \\ -\frac{iV}{2\omega_r}\sin\omega_r & \cos\omega_r - \frac{i\Delta}{2\omega_r}\sin\omega_r \end{pmatrix}$$

Also, $\Psi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} \alpha(0) = \tilde{\alpha}(0) = 1 \\ \beta(0) = \tilde{\beta}(0) = 0 \end{array}$

$$\Rightarrow \begin{pmatrix} \tilde{\alpha}(t) \\ \tilde{\beta}(t) \end{pmatrix} = \begin{pmatrix} \cos\omega_r + \frac{i\Delta}{2\omega_r}\sin\omega_r \\ -\frac{iV}{2\omega_r}\sin\omega_r \end{pmatrix}$$

$$\begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} = \begin{pmatrix} e^{-\frac{i}{2}(E_b+E_a-\omega)t} & \\ & e^{-\frac{i}{2}(E_b+E_a+\omega)t} \end{pmatrix} \begin{pmatrix} \cos\omega_r + \frac{i\lambda}{2\omega_r}\sin\omega_r \\ -\frac{iV}{2\omega_r}\sin\omega_r \end{pmatrix}$$

$$\boxed{\begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} = \begin{pmatrix} e^{-\frac{i}{2}(E_b+E_a-\omega)t}\left(\cos\omega_r + \frac{i\lambda}{2\omega_r}\sin\omega_r\right) \\ e^{-\frac{i}{2}(E_b+E_a+\omega)t}\left(-\frac{iV}{2\omega_r}\sin\omega_r\right) \end{pmatrix}}$$

Using $\alpha(t) = C_a(t)e^{-iE_at}$,

$\beta(t) = C_b(t)e^{-iE_bt}$

gives the same answer as the 1st method.

C. For $\omega = E_b - E_a$, $\omega_r = V$,

$|\beta(t)|^2 = 1 \Rightarrow \sin^2\left(\frac{V}{2}t\right) = 1$

$\Rightarrow \frac{V}{2}t = \frac{\pi}{2} + n\pi = \frac{\pi}{2}(1+2n) \therefore t = \frac{\pi}{V}(2n+1)$.

d. $|\beta|^2 = \frac{V^2}{4\omega_r^2}\sin^2(\omega_r t) = \frac{V^2}{\lambda^2+V^2}\sin^2\left(\frac{1}{2}\sqrt{\lambda^2+V^2}\,t\right)$

$\leq \frac{V^2}{\lambda^2+V^2}$

This value is always less than one unless $\lambda^2 = 0$, i.e.

$\omega = E_b - E_a$ = the resonant frequency.

e. In the interaction picture,

$$\psi_I(t) = T\exp\left\{-i\int_0^t V_I(t')\,dt'\right\}\psi_I(0)$$

$$= \psi_I(0) - i\int_0^t V_I(t')\,dt'\ \psi_I(0) + \text{higher order.}$$

To first order in perturbation, $\psi_I(t) = \left(1 - i\int_0^t V_I(t')dt'\right)\psi_I(0)$

Let $H_r = \begin{pmatrix} E_a & \\ & E_b \end{pmatrix} + \frac{V}{2}\begin{pmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & 0 \end{pmatrix}$

$$= H_0 + V(t),$$

then $V_I(t) = e^{iH_0 t}\, V(t)\, e^{-iH_0 t}$

$$= \begin{pmatrix} e^{iE_a t} & \\ & e^{iE_b t} \end{pmatrix} \frac{V}{2}\begin{pmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & 0 \end{pmatrix}\begin{pmatrix} e^{-iE_a t} & \\ & e^{-iE_b t} \end{pmatrix}$$

$$= \frac{V}{2}\begin{pmatrix} 0 & e^{-i(E_b - E_a - \omega)t} \\ e^{i(E_b - E_a - \omega)t} & 0 \end{pmatrix} = \frac{V}{2}\begin{pmatrix} 0 & e^{-i\Delta t} \\ e^{i\Delta t} & 0 \end{pmatrix}$$

$$\int_0^t V_I(t')\,dt' = \frac{V}{2}\begin{pmatrix} 0 & \frac{i}{\Delta}\left(e^{-i\Delta t} - 1\right) \\ -\frac{i}{\Delta}\left(e^{i\Delta t} - 1\right) & 0 \end{pmatrix}$$

$$= \frac{iV}{\Delta}\begin{pmatrix} 0 & e^{-i\frac{\Delta}{2}t}\left(\frac{e^{-i\frac{\Delta}{2}t} - e^{i\frac{\Delta}{2}t}}{2}\right) \\ -e^{i\frac{\Delta}{2}t}\left(\frac{e^{i\frac{\Delta}{2}t} - e^{-i\frac{\Delta}{2}t}}{2}\right) & \end{pmatrix}$$

$$\doteq \frac{V}{\lambda} \sin \frac{\lambda}{2} t \begin{pmatrix} 0 & e^{-i\frac{\lambda}{2}t} \\ e^{i\frac{\lambda}{2}t} & 0 \end{pmatrix}$$

$$\Rightarrow \psi_I(t) = \left[1 - i\frac{V}{\lambda} \sin \frac{\lambda}{2} t \begin{pmatrix} 0 & e^{-i\frac{\lambda}{2}t} \\ e^{i\frac{\lambda}{2}t} & 0 \end{pmatrix} \right] \psi_I(0)$$

$$\psi_S(t) = e^{-iH_0 t}\, \psi_I(t) \quad , \quad \psi_S(0) = \psi_I(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \left\{ \begin{pmatrix} e^{-iE_a t} & \\ & e^{-iE_b t} \end{pmatrix} - i\frac{V}{\lambda} \sin \frac{\lambda}{2} t \begin{pmatrix} 0 & e^{-i(\frac{\lambda}{2}+E_a)t} \\ e^{i(\frac{\lambda}{2}-E_b)t} & 0 \end{pmatrix} \right\} \psi_S(0)$$

$$\psi_S(t) = \begin{pmatrix} e^{-iE_a t} \\ 0 \end{pmatrix} - i\frac{V}{\lambda} \sin \frac{\lambda}{2} t \begin{pmatrix} 0 \\ e^{i(\frac{\lambda}{2}-E_b)t} \end{pmatrix}$$

$$= e^{-iE_a t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - i\frac{V}{\lambda} \sin \frac{\lambda}{2} t \; e^{i(\frac{\lambda}{2}-E_b)t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Probability to be in $|b\rangle = \dfrac{V^2}{\lambda^2} \sin^2 \dfrac{\lambda}{2} t$.

Note here that $|\alpha(t)|^2 + |\beta(t)|^2 = 1 + \dfrac{V^2}{\lambda^2} \sin^2 \dfrac{\lambda}{2} t \geq 1$

This is due to the fact that our calculation is 1st order only. This result will be a good approximation

if $\dfrac{V^2}{\lambda^2} \sin^2 \dfrac{\lambda}{2} t \ll 1$, or since $\sin^2 \frac{\lambda}{2} t \leq 1$,

$$\frac{V^2}{\lambda^2} = \frac{V^2}{(E_b - E_a - \omega)^2} \ll 1 .$$

Comparing the first order result

$$|c_b(t)|^2 = \frac{V^2}{\Delta^2} \sin^2 \frac{\Delta}{2} t$$

with the exact result

$$|c_b(t)|^2 = \frac{V^2}{\Delta^2 + V^2} \sin^2\left(\frac{1}{2}\sqrt{\Delta^2+V^2}\, t\right)$$

we can see that the first order result is the limit of the exact answer when we take $\frac{V^2}{\Delta^2} \ll 1$.

To first order in $\frac{V^2}{\Delta^2}$,

$$|c_b^{exact}(t)|^2 = \frac{V^2}{\Delta^2\left(1+\frac{V^2}{\Delta^2}\right)} \sin^2\left(\frac{\Delta}{2}\sqrt{1+\frac{V^2}{\Delta^2}}\, t\right)$$

$$\approx \frac{V^2}{\Delta^2} \sin^2\left(\frac{\Delta}{2} t\right).$$