# Solution Set 4

1. (a)

$$i \frac{\partial}{\partial t} \psi = -\frac{1}{2m} \nabla^2 \psi$$

Laplace transforming, we have

$$\int_0^\infty dt \; e^{i\omega t} \; i \frac{\partial}{\partial t} \psi(x,t) = -\frac{1}{2m} \nabla^2 \tilde{\psi}(x,\omega),$$

$$i \int_0^\infty dt \; \frac{\partial}{\partial t}\left(e^{i\omega t} \psi(x,t)\right) - i \int_0^\infty dt \; (i\omega) \, e^{i\omega t} \psi(x,t)$$

$$+ \frac{1}{2m} \nabla^2 \tilde{\psi}(x,\omega) = 0$$

$$\Rightarrow 0 - i\psi(x,0) + \omega \tilde{\psi}(x,\omega) + \frac{1}{2m} \nabla^2 \tilde{\psi}(x,\omega) = 0$$

$$-i \, \delta^3(\vec{x}) + \left(\omega + \frac{1}{2m}\nabla^2\right) \tilde{\psi}(x,\omega) = 0$$

In Fourier space,

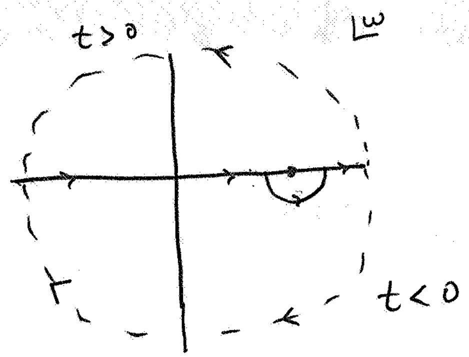
$$-i \int d^3x \; e^{-i\vec{x}\cdot\vec{p}} \, \delta^3(\vec{x}) + \int d^3x \; e^{-i\vec{x}\cdot\vec{p}} \left(\omega + \frac{1}{2m}\nabla^2\right) \tilde{\psi}(x,\omega) = 0$$

$$\Rightarrow \quad -i + \left(\omega - \frac{p^2}{2m}\right) \tilde{\psi}(\vec{p},\omega) = 0$$

$$\Rightarrow \quad \tilde{\psi}(\vec{p},\omega) = \frac{i}{\omega - p^2/2m}$$

Then, $\tilde{\psi}(\vec{p},t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{\psi}(\vec{p},\omega)$

$$= \frac{i}{2\pi} \int_{-\infty}^{\infty} d\omega \, e^{-i\omega t} \frac{1}{\omega - p^2/2m}$$



By going under the pole, we ensure that for $t<0$ $\tilde{\psi}(\vec{p},t) = 0$. In fact for $t<0$ we can close the contour in the bottom half plane, and since $\tilde{\psi}(\vec{p},\omega)$ has no poles the integral gives 0.

For $t>0$ we close the contour on the top half plane and pick up the residue of the pole at $\omega = p^2/2m$.

$$\tilde{\psi}(\vec{p},t) = \begin{cases} 0, & t<0 \\ -e^{-ip^2/2m\, t}, & t>0 \end{cases}$$

$$\psi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \tilde{\psi}(\vec{p},t)$$

$$= \frac{1}{(2\pi)^2} \int_0^\infty p^2 dp \int_{-1}^{1} d(\cos\theta) \, e^{ipx\cos\theta} \, e^{-i p^2/2m\, t}$$

$$= \frac{1}{(2\pi)^2} \int_0^\infty p^2 dp \; e^{-ip^2/2m\,t} \; \frac{1}{ipx} \left( e^{ipx} - e^{-ipx} \right)$$

$$= \frac{-i}{(2\pi)^2} \frac{1}{x} \left(-i \frac{d}{dx}\right) \int_0^\infty dp \; e^{-ip^2/2m\,t} \left( e^{ipx} + e^{-ipx} \right)$$

$$= -\frac{1}{2\pi^2} \frac{1}{x} \frac{d}{dx} \int_{-\infty}^\infty dp \; e^{-ip^2/2m\,t} \; e^{ipx}$$

$$\Big\| \; -i \frac{p^2}{2m} t + ipx = -\frac{i}{2m} t \left( p^2 - \frac{2mx}{t} p \right)$$

$$= -\frac{i}{2m} t \left( p - \frac{mx}{t} \right)^2 + \frac{it}{2m} \left( \frac{mx}{t} \right)^2$$

$$= \frac{-i}{2m} t \left( p - \frac{mx}{t} \right)^2 + \frac{i}{2} \frac{mx^2}{t} \; \Big\|$$

$$-\frac{1}{2\pi^2} \frac{1}{x} \frac{d}{dx} \; e^{\frac{i}{2}\frac{mx^2}{t}} \int_{-\infty}^\infty dp' \; e^{\frac{-i}{2m} t (p')^2}$$

$$= -\frac{1}{2\pi^2} \frac{1}{x} \left( \frac{imx}{t} \right) e^{\frac{i}{2} mx^2/t} \sqrt{\frac{2m}{it} \pi}$$

$$\psi(x,t) = \frac{1}{\sqrt{2}} \left( -\frac{m^2}{t^2 \pi^2} \right)^{\frac{3}{4}} e^{\frac{i}{2} mx^2/t}$$

(b) $\frac{1}{\tau^2}\left[ x(t+2\tau) - 2x(t+\tau) + x(t)\right] = -\Omega^2 x(t)$

$$x(t+2\tau) - 2x(t+\tau) + x(t) = -\tau^2\Omega^2 x(t)$$

Laplace tranform both sides: First the LHS

$$\int_0^\infty e^{-st}dt\, x(t+2\tau) - 2\int_0^\infty e^{-st}dt\, x(t+\tau) + \int_0^\infty e^{-st}dt\, x(t)$$

Let $t_1 = t+2\tau$ , $t_2 = t+\tau$

$$\int_{2\tau}^\infty e^{-s(t_1-2\tau)}dt_1\, x(t_1) - 2\int_\tau^\infty e^{-s(t_2-\tau)}dt_2\, x(t_2) + \int_0^\infty e^{-st}dt\, x(t)$$

$$= \int_0^\infty e^{-s(t_1-2\tau)}x(t_1)dt_1 - \int_0^{2\tau} e^{-s(t_1-2\tau)}x(t_1)dt_1$$

$$-2\int_0^\infty e^{-s(t_2-\tau)}x(t_2)\,dt_2 + 2\int_0^\tau e^{-s(t_2-\tau)}x(t_2)\,dt_2 + \int_0^\infty e^{-st}dt\, x(t)$$

$$= \int_0^\infty e^{-st}x(t)dt\left(e^{2s\tau} - 2e^{s\tau}+1\right) - e^{2s\tau}\int_0^{2\tau} e^{-st_1}x(t_1)dt_1$$

$$+ 2e^{s\tau}\int_0^\tau e^{-st_2}x(t_2)dt_2$$

$$= \left(e^{2s\tau} - 2e^{s\tau}+1\right)\tilde{x}(s) - e^{2s\tau}X\int_0^{2\tau} e^{-st_1}dt_1 + 2e^{s\tau}X\int_0^\tau e^{-st_2}dt_2$$

$$= \left(e^{2s\tau} - 2e^{s\tau}+1\right)\tilde{x}(s) - e^{2s\tau}X\left(\frac{-1}{s}\right)\left[e^{-st_1}\right]_0^{2\tau} + 2e^{s\tau}X\left(\frac{-1}{s}\right)\left[e^{-st_2}\right]_0^\tau$$

$$= \left(e^{2s\tau} - 2e^{s\tau} + 1\right)\tilde{x}(s) + \frac{X}{s} e^{2s\tau}\left(e^{-2s\tau} - 1\right) - \frac{2X}{s} e^{s\tau}\left(e^{-s\tau} - 1\right)$$

$$= \left(e^{2s\tau} - 2e^{s\tau} + 1\right)\tilde{x}(s) + \frac{X}{s} e^{s\tau}\left(e^{-s\tau} - e^{s\tau} - 2e^{-s\tau} + 2\right)$$

$$= \left(e^{2s\tau} - 2e^{s\tau} + 1\right)\tilde{x}(s) + \frac{X}{s}\left(-1 - e^{2s\tau} + 2e^{s\tau}\right)$$

$$= \left(e^{2s\tau} - 2e^{s\tau} + 1\right)\left(\tilde{x}(s) - \frac{X}{s}\right)$$

$$\text{RHS} = \Omega^2\tau^2\,\tilde{x}(s)$$

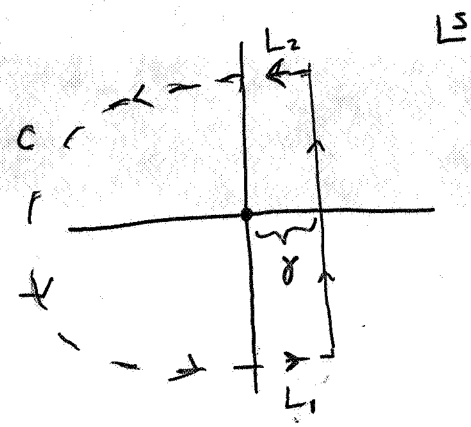
$$\Rightarrow \left(e^{2s\tau} - 2e^{s\tau} + 1 + \Omega^2\tau^2\right)\tilde{x}(s) = \frac{X}{s}\left(e^{2s\tau} - 2e^{s\tau} + 1\right)$$

$$\tilde{x}(s) = \frac{X}{s}\,\frac{\left(e^{2s\tau} - 2e^{s\tau} + 1\right)}{\left(e^{2s\tau} - 2e^{s\tau} + 1 + \Omega^2\tau^2\right)}$$

$$= \frac{X}{s}\,\frac{\left(e^{s\tau} - 1\right)^2}{\left[\left(e^{s\tau} - 1\right)^2 + \Omega^2\tau^2\right]}$$

$$= \frac{X}{s} \quad \text{for } \Omega\tau \ll 1$$

$$x(t) = \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st}\, \tilde{x}(s)\, ds = \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st}\, \frac{X}{s}\, ds$$




For $t < 0$ we close the contour from the right, there are no poles, therefore the integral is zero.

$$x(t) = 0 \ , \ t < 0$$

For $t > 0$ we close the contour from the left, pick up the residue from the pole at the origin.

$$\oint_c e^{st} \frac{X}{s}\, ds = \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} \frac{X}{s}\, ds + \int_{L_2} e^{st} \frac{X}{s}\, ds + \int_{\substack{\text{semi}\\ \text{circle}}} e^{st} \frac{X}{s}\, ds + \int_{L_1} e^{st} \frac{X}{s}\, ds$$

Since $t > 0$, for any $s$ with $Re(s) < 0$ $\quad e^{-st} \to 0$.

so the contribution from the semi circle $\to 0$

At $L_2$ $s$ has small real and large imaginary parts

Let $s = x + iy$ , $\dfrac{e^{st}}{s} = \dfrac{e^{x+iy}}{x+iy} \to \dfrac{e^{x} e^{iy}}{iy}$ for $|y| \gg x$

$$\to \frac{(1+x)\, e^{iy}}{iy} \to 0 \quad \text{as } y \to \infty .$$

The same holds for $L_1$.

So, $\oint e^{st} \frac{X}{s} ds = 2\pi i (X) = \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} \frac{X}{s} ds$

$\Rightarrow X(t) = X \quad , \quad t > 0$

This result is trivial. We got it because we discarded the $\Omega^2 \tau^2$ term entirely. Let's repeat the calculation keeping terms to first order in $\Omega^2 \tau^2$.

$$\tilde{X}(s) = \frac{X}{s} \frac{(e^{s\tau} - 1)^2}{[(e^{s\tau} - 1)^2 + \Omega^2 \tau^2]}$$

has poles at $e^{s\tau} - 1 = \pm i \Omega \tau$

$$\Rightarrow e^{s\tau} = \pm i \Omega \tau + 1$$

$$s = \frac{1}{\tau} \log [1 \pm i \Omega \tau]$$

$$= \frac{1}{\tau} \left\{ \pm i \Omega \tau - \frac{1}{2} (\pm i \Omega \tau)^2 + \cdots \right\}$$

$$= \pm i \Omega + \frac{1}{2} \Omega^2 \tau + \cdots$$

$$= \pm i \Omega + \frac{1}{2} \Omega^2 \tau \quad \text{for small } \Omega \tau$$

Notice that the pole has positive real component. When we inverse Laplace transform

$x(t) = \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} \tilde{x}(s) ds \propto e^{(\frac{1}{2} \Omega^2 \tau) t}$ which

is unstable!

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2.

(a) Matrix elements of $H$ are:

$$\langle \vec{r} | H | \vec{r}' \rangle = \langle \vec{r} | -\frac{\nabla^2}{2m} | \vec{r}' \rangle = -\frac{\nabla^2}{2m} \delta^3(\vec{r} - \vec{r}')$$

$$\langle \vec{r} | H | a \rangle = \lambda r^a \chi(r) \quad , \quad r^a = (x, y, z)^a$$

$$\langle a | H | \vec{r} \rangle = \lambda r^a \chi^*(r)$$

$$\langle a | H | b \rangle = \delta_{ab} E_0$$

An eigenstate $|\psi\rangle$ of $H$ can be decomposed into a linear combination of the above states:

$$|\psi\rangle = \int d^3r \; \psi(\vec{r}) \, |\vec{r}\rangle + \sum_a \alpha^a |a\rangle \qquad (1)$$

$$H|\psi\rangle = \int d^3r \; \psi(\vec{r}) \, H|\vec{r}\rangle + \sum_a \alpha^a H|a\rangle \qquad (2)$$

Dotting (2) with a state $\langle r'|$ we get

$$\langle \vec{r}' | H | \psi \rangle = \int d^3r \; \psi(\vec{r}) \langle \vec{r}' | H | \vec{r} \rangle + \sum_a \alpha^a \langle \vec{r}' | H | a \rangle$$

$\|$ $\quad H|\psi\rangle = E|\psi\rangle \; , \quad \langle r' | \psi \rangle = \psi(\vec{r}') \, ,$

$$\int d^3r \; \psi(r) \langle \vec{r}' | H | \vec{r} \rangle = \int d^3r \; \psi(\vec{r}) \left( -\frac{\nabla^2}{2m} \delta^3(\vec{r} - \vec{r}') \right)$$

After integrating by parts twice,

$$= \int d^3r \left( -\frac{\nabla^2}{2m} \psi(\vec{r}) \right) \delta^3(\vec{r} - \vec{r}')$$

$$= -\frac{\nabla'^2}{2m} \psi(\vec{r}') \; , \quad \nabla' \text{ is wrt } \vec{r}' \; \|$$

$$\therefore E\psi(\vec{r}') = -\frac{\nabla^2}{2m}\psi(\vec{r}') + \lambda \sum_a r'^a \alpha^a \chi(\vec{r}')$$

$$\left(E + \frac{\nabla^2}{2m}\right)\psi(\vec{r}') = \lambda r'^a \alpha^a \chi(\vec{r}') , \qquad (3)$$

From now on we will use Einstein's summation convention $r'^a \alpha^a = \sum_a r'^a \alpha^a$

Now dotting (2) with $\langle b|$,

$$\langle b|H|\psi\rangle = \int d^3r\, \psi(\vec{r})\, \langle b|H|\vec{r}\rangle + \sum_a \alpha^a \langle b|H|a\rangle$$

$$E\langle b|\psi\rangle = \int d^3r\, \psi(\vec{r})\, \lambda r^b \chi^*(r) + \alpha^a \delta^{ab} E_0$$

$$\| \ \langle b|\psi\rangle = \int d^3r\, \psi(\vec{r})\, \underbrace{\langle b|\vec{r}\rangle}_{0} + \sum_a \alpha^a \langle b|a\rangle$$

$$= \alpha^a \delta^{ab}$$

$$= \alpha^b \ \|$$

$$E\alpha^b = \int d^3r\, \lambda \psi(\vec{r}) r^b \chi^*(r) + \alpha^b E_0$$

$$\alpha^b = \frac{\lambda}{E - E_0} \int d^3r\, \psi(\vec{r})\, r^b \chi^*(r) \qquad (4)$$

(4) → (3)

$$\left(E + \frac{\nabla^2}{2m}\right)\psi(\vec{r}') = \frac{\lambda^2}{E-E_0} \chi(\vec{r}')\, r'^a \int d^3r\, \psi(\vec{r})\, r^a \chi^*(\vec{r})$$

$$(-2mE - \nabla^2)\, \psi(\vec{r}') = -\frac{2m\lambda^2}{E-E_0} \chi(\vec{r}')\, r'^a \int d^3r\, \psi(\vec{r})\, r^a \chi^*(\vec{r})$$

$$(-k^2 + \underset{\sim}{p}^2)\,\psi(\vec{r}') = \frac{-2m d^2}{E - E_0}\,\chi(\vec{r}')\,r'^a \int d^3r\,\psi(\vec{r})\,r^a\,\chi^*(\vec{r})$$

Let $\psi(\vec{r}') = e^{ikz'} + F(\vec{r}')$, then

$$(-k^2 + \underset{\sim}{p}^2)\,\psi(\vec{r}') = \underbrace{(-k^2 + \underset{\sim}{p}^2)\,e^{ikz'}}_{0} + (-k^2 + \underset{\sim}{p}^2)\,F(\vec{r}')$$

$$= (-k^2 + \underset{\sim}{p}^2)\,F(\vec{r}')$$

$$\boxed{(-k^2 + \underset{\sim}{p}^2)\,F(\vec{r}') = -\frac{2m d^2}{E - E_0}\,\chi(\vec{r}')\,r'^a \int d^3r\,\psi(\vec{r})\,r^a\,\chi^*(\vec{r})}$$

Fourier transforming, we get

$$(-k^2 + p^2)\,\tilde{F}(\vec{p}) = -\frac{2m d^2}{E - E_0}\int d^3r'\,e^{-i\vec{p}\cdot\vec{r}'}\,\chi(\vec{r}')\,r'^a \int d^3r\,\psi(\vec{r})\,r^a\,\chi^*(\vec{r})$$

$$= \frac{-2m d^2}{E - E_0}(i)\frac{\partial}{\partial p^a}\tilde{\chi}(\vec{p}) \int d^3r\,\psi(\vec{r})\,r^a\,\chi^*(\vec{r})$$

We have used $\displaystyle\int d^3r'\,e^{-i\vec{p}\cdot\vec{r}'}\,r'^a\,\chi(\vec{r}')$

$$= i\frac{\partial}{\partial p^a}\int \frac{d^3\vec{p}}{(2\pi)^3}\,e^{i\vec{p}\cdot\vec{r}'}\,\chi(\vec{r}')$$

$$= i\frac{\partial}{\partial p^a}\tilde{\chi}(\vec{p})\ ; \quad \tilde{\chi}(\vec{p}) = \tilde{\chi}(p) \text{ by spherical symmetry}$$

$$= i\frac{\partial}{\partial p}\frac{\partial p}{\partial p^a}\tilde{\chi}(p) = i\frac{p^a}{p}\frac{d}{dp}\tilde{\chi}(p)$$

$$(-k^2+p^2)\,\tilde{F}(\vec{p}) = \frac{-2im\lambda^2}{E-E_0}\,\frac{p^a}{p}\frac{d}{dp}\tilde{\chi}(p)\int d^3r\, r^a\,\psi(\vec{r})\,\chi^*(\vec{r}) \quad (5)$$

$$\int d^3r\, r^a\,\psi(\vec{r})\,\chi^*(\vec{r}) = \int d^3r\, r^a\,\chi^*(\vec{r})\int\frac{d^3q}{(2\pi)^3}\,e^{i\vec{q}\cdot\vec{r}}\,\tilde{\psi}(\vec{q})$$

$$= \int\frac{d^3q}{(2\pi)^3}\,\tilde{\psi}(\vec{q})\int d^3r\, r^a\,e^{i\vec{q}\cdot\vec{r}}\,\chi^*(\vec{r})$$

$$= \int\frac{d^3q}{(2\pi)^3}\,\tilde{\psi}(\vec{q})\left[i\frac{\partial}{\partial q^a}\int d^3r\,e^{-i\vec{q}\cdot\vec{r}}\,\chi(\vec{r})\right]^*$$

$$= \int\frac{d^3q}{(2\pi)^3}\,\tilde{\psi}(\vec{q})\left(-i\frac{\partial}{\partial q^a}\tilde{\chi}^*(q)\right)$$

$$= -i\int\frac{d^3q}{(2\pi)^3}\,\tilde{\psi}(\vec{q})\,\frac{q^a}{q}\frac{d}{dq}\tilde{\chi}^*(q)$$

$$\Big\|\quad \psi(\vec{r}) = e^{ik\hat{z}\cdot\vec{r}} + F(\vec{r})$$

$$\Rightarrow\quad \tilde{\psi}(\vec{q}) = (2\pi)^3\,\delta^3(q-k\hat{z}) + \tilde{F}(\vec{q}) \quad \Big\|$$

So,
$$-i\int\frac{d^3q}{(2\pi)^3}\left((2\pi)^3\delta^3(q-k\hat{z}) + \tilde{F}(\vec{q})\right)\frac{q^a}{q}\frac{d}{dq}\tilde{\chi}^*(q)$$

$$= -i\,\frac{(k\hat{z})^a}{k}\frac{d}{dk}\tilde{\chi}^*(k) - i\int\frac{d^3q}{(2\pi)^3}\,\tilde{F}(\vec{q})\,\frac{q^a}{q}\frac{d}{dq}\tilde{\chi}^*(q)$$

$$= -i(\hat{z})^a\frac{d}{dk}\tilde{\chi}^*(k) - i\,\Phi^a, \quad \text{where}$$

$$\Phi^a = \int\frac{d^3q}{(2\pi)^3}\,\tilde{F}(\vec{q})\,\frac{q^a}{q}\frac{d}{dq}\tilde{\chi}^*(q) \quad (6)$$

From (5),

$$\tilde{F}(\vec{p}) = \frac{-2md^2}{E-E_0} \frac{1}{p^2-k^2} \frac{d}{dp}\tilde{x}(p)\left[(\hat{p})^b(\hat{z})^b \frac{d}{dk}\tilde{x}^*(k) + (\hat{p})^b \Phi^b\right] \qquad (7)$$

(7) → (6),

$$\Phi^a = \int \frac{d^3q}{(2\pi)^3} \tilde{F}(\vec{q})\,(\hat{q})^a \frac{d}{dq}\tilde{X}^*(q)$$

Be careful with indices here, they don't have to be the same!

$$= \frac{-2md^2}{E-E_0}\int \frac{d^3q}{(2\pi)^3}\frac{1}{q^2-k^2}\left|\frac{d}{dq}\tilde{X}(q)\right|^2 \hat{q}^a \left[\hat{q}^b \hat{z}^b \tilde{X}^*(k) + \hat{q}^b \Phi^b\right]$$

$$= \frac{-2md^2}{E-E_0}\int \frac{d^3q}{(2\pi)^3}\frac{1}{q^2-k^2}\left|\frac{d}{dq}\tilde{X}(q)\right|^2 \hat{q}^a \hat{q}^b\left(\tilde{X}^*(k)\hat{z}^b + \Phi^b\right)$$

$$= \frac{-2md^2}{E-E_0}\,\underline{X}(k)^{ab}\left(\tilde{X}^*(k)\hat{z}^b + \Phi^b\right) \quad \text{, where}$$

$$\underline{X}^{ab}(k) = \int \frac{d^3q}{(2\pi)^3}\frac{1}{q^2-k^2}\left|\frac{d}{dq}\tilde{X}(q)\right|^2 \hat{q}^a \hat{q}^b$$

$$\left[\delta^{ab} + \frac{2md^2}{E-E_0}\underline{X}^{ab}\right]\Phi^b = \frac{-2md^2}{E-E_0}\tilde{X}^*(k)\,\underline{X}^{ab}\hat{z}^b \qquad (8)$$

This is a matrix equation that we can invert to find $\Phi^a$. First let's evaluate $\underline{X}^{ab}$.

Note that when $a \neq b$ $\Sigma^{ab}$ evaluates to 0 by spherical symmetry. Eq. If $a=1$, $b=2$,

$$\Sigma^{12} = \int \frac{d^3q}{(2\pi)^3} \frac{1}{q^2-k^2} \left| \frac{d}{dq} \tilde{x}(q) \right|^2 \hat{q}^1 \hat{q}^2$$

The integrand is anti-symmetric under $\hat{q}^1 \to -\hat{q}^1$

$$\Rightarrow X^{12} = \int_{-\infty}^{\infty} \frac{dq^3}{2\pi} \int_{-\infty}^{\infty} \frac{dq^2}{2\pi} \hat{q}^2 \int_{-\infty}^{\infty} \frac{dq^1}{2\pi} \frac{1}{q^2-k^2} \left| \frac{d}{dq} \tilde{x}(q) \right|^2 \hat{q}^1 \to 0$$

$$\therefore \Sigma^{ab} = N \delta^{ab} \int \frac{d^3q}{(2\pi)^3} \frac{1}{q^2-k^2} \left| \frac{d}{dq} \tilde{x}(q) \right|^2 ,$$ N is normalization factor.

To determine N let's take the trace on both sides,

$$\mathrm{Tr}\, \Sigma^{ab} = \delta^{ab} \Sigma^{ab} \left( = \sum_{a,b} \delta^{ab} \Sigma^{ab} \right)$$

$$= \int \frac{d^3q}{(2\pi)^3} \frac{1}{q^2-k^2} \left| \frac{d}{dk} \tilde{x}(k) \right|^2 \underbrace{\hat{q}^a \hat{q}^b \delta^{ab}}_{=1}$$

$$= \int \frac{d^3q}{(2\pi)^3} \frac{1}{q^2-k^2} \left| \frac{d}{dk} \tilde{x}(k) \right|$$

$$= \Sigma(k)$$

On the other hand, $\mathrm{Tr} \left\{ N \delta^{ab} \int \frac{d^3q}{(2\pi)^3} \frac{1}{q^2-k^2} \left| \frac{d}{dk} \tilde{x}(k) \right|^2 \right\}$

$$= N \delta^{ab} \delta^{ab} \Sigma(k) = N(3) \Sigma(k) \Rightarrow N = 1/3$$

$$\therefore \Sigma^{ab} = \frac{\delta^{ab}}{3} \Sigma(k), \quad \Sigma(k) = \int \frac{d^3q}{(2\pi)^3} \frac{1}{q^2-k^2} \left| \frac{d}{dk} \tilde{x}(k) \right|^2$$

$$(8) \Rightarrow \left( 1 + \frac{2md^2}{3} \frac{\Sigma(k)}{E-E_0} \right) \delta^{ab} \Phi^b = -\frac{2md^2}{3} \frac{\Sigma(k)}{E-E_0} \tilde{x}^*(k) \delta^{ab} \hat{z}^b$$

$$\boxed{\Phi^a = -\frac{2md^2}{3} \frac{\Sigma(k) \tilde{x}^*(k)}{E-E_0 + \frac{2md^2}{3} \Sigma(k)} \hat{z}^a}$$

(10) → (7),

$$\tilde{F}(\vec{p}) = \frac{-\frac{2}{3}m\lambda^2}{E-E_0}\frac{1}{p^2-k^2}\frac{d}{dp}\tilde{\chi}(p)\,\hat{p}^b\hat{z}^b\left[\frac{d}{dk}\tilde{\chi}^*(k) + \frac{\frac{2}{3}m\lambda^2\frac{d}{dk}\tilde{\chi}^*(k)}{E-E_0+\frac{2}{3}m\lambda^2\Sigma(k)}\right]$$

$$= \frac{-2m\lambda^2}{\cancel{E-E_0}}\frac{1}{p^2-k^2}\frac{d}{dp}\tilde{\chi}(p)\,\hat{p}^b\hat{z}^b\left(\frac{\cancel{(E-E_0)}\frac{d}{dk}\tilde{\chi}^*(k)}{E-E_0+\frac{2}{3}m\lambda^2\Sigma(k)}\right)$$

Inverting the fourier transform, we get

$$F(\vec{r}) = -2m\lambda^2\frac{\frac{d}{dk}\tilde{\chi}^*(k)}{E-E_0+\frac{2}{3}m\lambda^2\Sigma(k)}\int\frac{d^3p}{(2\pi)^3}e^{i\vec{p}\cdot\vec{r}}\frac{1}{p^2-k^2}\hat{p}^b\hat{z}^b\frac{d}{dp}\tilde{\chi}(p)$$

$$\int\frac{d^3p}{(2\pi)^3}e^{i\vec{p}\cdot\vec{r}}\frac{1}{p^2-k^2}\hat{p}^b\hat{z}^b\frac{d}{dp}\tilde{\chi}(p)$$

$$= \int\frac{d^3p}{(2\pi)^3}e^{i\vec{p}\cdot\vec{r}}\frac{1}{p^2-k^2}\hat{z}^b\frac{\partial}{\partial p^a}\tilde{\chi}(p)$$

$$= \int\frac{d^3p}{(2\pi)^3}e^{i\vec{p}\cdot\vec{r}}\frac{1}{p^2-k^2}\hat{z}^b\frac{\partial}{\partial p^a}\int d^3r'\,e^{-i\vec{p}\cdot\vec{r}'}\chi(\vec{r}')$$

$$= -i\int d^3r'\,\hat{z}^b r'^b\chi(r')\int\frac{d^3p}{(2\pi)^3}e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}\frac{1}{p^2-k^2}$$

$$= -i\int d^3r'\,\hat{z}^b r'^b\chi(r')\frac{e^{ik|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}$$

$|\vec{r}| \gg |\vec{r}'|$

$$= -i\frac{e^{ikr}}{4\pi r}\int d^3r'\,\hat{z}^b r'^b\chi(r')\,e^{-ik\hat{r}\cdot\vec{r}'}$$

$$= -i \frac{e^{ikr}}{4\pi r} \hat{z}^b \frac{i\partial}{\partial(k\hat{r})^b} \tilde{\chi}(k)$$

$$= \frac{e^{ikr}}{4\pi r} \hat{z}^b \frac{k\hat{r}^b}{k} \frac{d}{dk}\tilde{\chi}(k)$$

$$= \frac{e^{ikr}}{4\pi r} \frac{d}{dk}\tilde{\chi}(k) \hat{z}^b \hat{r}^b$$

$$F(\vec{r}) = \frac{e^{ikr}}{r} \frac{-md^2}{2\pi} \frac{1}{E-E_0+\frac{2}{3}md^2\Sigma(k)} \left|\frac{d}{dk}\tilde{\chi}(k)\right|^2 \cos\theta$$

$$f_k = \frac{-md^2}{2\pi} \frac{1}{E-E_0+\frac{2}{3}md^2\Sigma(k)} \left|\frac{d}{dk}\tilde{\chi}(k)\right|^2 \cos\theta$$

$$\Sigma(k) = \int \frac{d^3q}{(2\pi)^3} \frac{1}{q^2-k^2} \left|\frac{d}{dk}\tilde{\chi}(k)\right|^2 = \Sigma(k) + iH(k)$$

$$= \frac{4\pi}{(2\pi)^3} \int_0^\infty dq \frac{q^2}{q^2-k^2} \left|\frac{d}{dk}\tilde{\chi}(k)\right|^2$$

$$= \frac{1}{2\pi^2} \int_0^\infty dq \frac{h(q)}{q-k}, \quad h(q) = \frac{q^2}{q+k}\left|\frac{d}{dq}\tilde{\chi}(k)\right|^2$$

$$= \frac{1}{2\pi^2}\left\{ P \int_0^\infty dq \frac{h(q)}{q-k} + i\pi h(k)\right\}, \text{ where}$$

we've moved the contour down

$$\Rightarrow H(k) = \frac{1}{2\pi} h(k) \, ; \quad \Sigma(k) = \frac{1}{2\pi^2} P \int_0^\infty dq \frac{h(q)}{q-k}$$

$$f_k = -\frac{m d^2}{2\pi} \frac{1}{E - E_0 + \frac{2}{3} m d^2 \Sigma(k) + i\frac{2}{3} m d^2 H(k)} \left| \frac{d}{dk} \tilde{\chi}(k) \right|^2 \cos\theta$$

$$\frac{2}{3} m d^2 H(k) = \frac{m d^2}{3\pi} h(k) = \frac{m d^2}{3\pi} \frac{k^2}{2k} \left| \frac{d}{dk} \tilde{\chi}(k) \right|^2$$

$$= \frac{1}{3} \frac{m d^2}{2\pi} k \left| \frac{d}{dk} \tilde{\chi}(k) \right|^2$$

$$= \frac{\Gamma_1}{2}$$

$$f_k = -\frac{m d^2}{2\pi} \frac{1}{E - E^* + i\Gamma_1/2} \left| \frac{d}{dk} \tilde{\chi}(k) \right|^2 \cos\theta \quad , E^* = E_0 - \frac{2}{3} m d^2 \Sigma(k)$$

$$= -\frac{1}{k} 3\, \Gamma_1/2 \frac{1}{E - E^* + i\Gamma_1/2} \cos\theta$$

$$= -\frac{1}{k} \frac{3(\Gamma_1/2)(E - E^* - i\Gamma_1/2)}{(E - E^*)^2 + (\Gamma_1/2)^2} \cos\theta$$

$$= \underline{\frac{1}{k} e^{i\delta_1} \sin\delta_1 \, 3\cos\theta} \quad , \quad \cos\delta_1 = \frac{E - E^*}{\left((E - E^*)^2 + (\Gamma_1/2)^2\right)^{\frac{1}{2}}}$$

$$\sin\delta_1 = \frac{\Gamma_1/2}{\left[(E - E^*)^2 + (\Gamma_1/2)^2\right]^{\frac{1}{2}}}$$

b) Matrix elements are:

$$\langle \vec{r}, m | H | \vec{r}\,'; m' \rangle = \langle \vec{r} | -\frac{\nabla^2}{2m} | \vec{r}\,' \rangle \langle m | m' \rangle = -\frac{\nabla^2}{2m} \delta^3(\vec{r}-\vec{r}\,') \, \delta_{mm'}$$

$$\langle \vec{r}, m | H | M \rangle = \lambda (\sigma^a)_{mM} \, r^a \, \chi(r)$$

$$\langle M | H | m, \vec{r} \rangle = \lambda (\sigma^a)^*_{mM} \, r^a \, \chi^*(r)$$

$$= \lambda (\sigma^{a\dagger})_{Mm} \, r^a \, \chi^*(r)$$

$$= \lambda (\sigma^a)_{Mm} \, r^a \, \chi^*(r) \quad \text{since } (\sigma^a)^\dagger = \sigma^a$$

$$\langle M | H | M' \rangle = E_0 \, \delta_{MM'}$$

Let $|\psi, m\rangle = \int d^3r \; \psi_m(\vec{r}) \, |\vec{r}, m\rangle + \sum_M \alpha_M |M\rangle$, (1)

then $\langle r'; m' | \psi, m \rangle = \int d^3r \, \psi_m(\vec{r}) \, \langle \vec{r}\,'; m' | \vec{r}, m \rangle + 0$

$$= \int d^3r \, \psi_m(\vec{r}) \, \delta^3(\vec{r}-\vec{r}\,') \, \delta_{m'm}$$

$$= \psi_{m'}(\vec{r}\,') \tag{2}$$

$$\langle M' | \psi, m \rangle = 0 + \sum_M \alpha_M \langle M' | M \rangle = \alpha_M \delta_{M'M}$$

$$= \alpha_{M'} \tag{3}$$

$$H|\psi, m\rangle = \int d^3r' \, \psi_{m'}(r') \, H |\vec{r}\,', m'\rangle + \sum_M \alpha_M H |M\rangle$$

$$\langle \vec{r}, m | H | \psi, m' \rangle = -\frac{\nabla^2}{2m} \psi_m(r) + \lambda (\sigma^a)_{mM} \, r^a \, \alpha_M \, \chi(r)$$

$$\left(E + \frac{\nabla^2}{2m}\right) \psi_m(r) = \lambda \, \chi(r) \, r^a \, \sigma^a_{mM} \, \alpha_M$$

$$\left(-k^2 + \underset{\sim}{p}^2\right) \psi_m(r) = -2m\lambda \, \chi(r) \, r^a \, \sigma^a_{mM} \, \alpha_M \tag{4}$$

$$\langle M'|H|\psi, m'\rangle = \int d^3r'\, \psi_{m'}(r')\, \langle M'|H|\vec{r}', m'\rangle + \sum_M \alpha_M \langle M'|H|M\rangle$$

$$E\alpha_{M'} = \int d^3r'\, \psi_{m'}(r')\, \lambda\, \chi^*(\vec{r}')\, r^a \sigma^a_{M'm'} + E_0 \alpha_{M'}$$

$$\alpha_M = \frac{\lambda}{E-E_0} \int d^3r\, \psi_m(\vec{r})\, \chi^*(r)\, r^a \sigma^a_{Mm} \qquad (5)$$

$\psi_m(\vec{r}) = \xi_m e^{ikz} + F_m(\vec{r})$ , where $\xi_m$ are the spin up or down $S_z$ eigenvectors, $\xi_+ = \begin{pmatrix}1\\0\end{pmatrix}$, $\xi_- = \begin{pmatrix}0\\1\end{pmatrix}$.

|| Note: by $(-k^2 + \underset{\sim}{p}^2)\,\psi_m(\vec{r})$ what we mean is $(-k^2 + \underset{\sim}{p}^2)\,\delta_{mm'}\,\psi_{m'}(\vec{r})$ , and $(-k^2 + \underset{\sim}{p}^2)\,\xi_m e^{ikz} = 0$ ||

$$(-k^2 + \underset{\sim}{p}^2)\,\psi_m(\vec{r}) = (-k^2 + \underset{\sim}{p}^2)\, F_m(\vec{r})$$

Fourier transforming,

$$(-k^2 + p^2)\, \tilde{F}_m(\vec{p}) = -2m\lambda i\, \hat{p}^a \frac{d}{dp} \tilde{\chi}(p)\, \sigma^a_{mM}\, \alpha_M \qquad (6)$$

|| We've put the fourier transform of $r^a \chi(\vec{r}) = \frac{p^a}{p}\frac{d}{dp}\tilde{\chi}(p) = \hat{p}^a \frac{d}{dp}\tilde{\chi}(p)$ as in (a). ||

In Fourier space (5) becomes

$$\alpha_M = \frac{-i\lambda}{E-E_0} \int \frac{d^3p}{(2\pi)^3}\, \tilde{\psi}_m(p)\, \hat{p}^a \frac{d}{dp} \tilde{\chi}^*(p)\, \sigma^a_{Mm} \qquad (7)$$

(7) → (6),

$$\tilde{F}_{m'}(\vec{q}) = \frac{-2m\lambda^2}{E-E_0} \frac{\hat{q}^a \sigma^a_{m'M} \frac{d}{dq}\tilde{\chi}(q)}{q^2-k^2} \int \frac{d^3p}{(2\pi)^3} \tilde{\psi}_m(p)\, \hat{p}^b \sigma^b_{Mm} \frac{d}{dp}\tilde{\chi}^*(p)$$

$$\tilde{\psi}_m(p) = \xi_m (2\pi)^3 \delta^3(\vec{p}-k\hat{z}) + \tilde{F}_m(\vec{p})$$

$$\Rightarrow \int \frac{d^3p}{(2\pi)^3} \tilde{\psi}_m(p)\, \hat{p}^b \sigma^b_{Mm} \frac{d}{dp}\tilde{\chi}^*(p)$$

$$= \xi_m \hat{z}^b \sigma^b_{Mm} \frac{d}{dk}\tilde{\chi}^*(k) + \int \frac{d^3p}{(2\pi)^3} \tilde{F}_m(\vec{p})\, \hat{p}^b \sigma^b_{Mm} \frac{d}{dp}\tilde{\chi}^*(p)$$

$$= \xi_m \hat{z}^b \sigma^b_{Mm} \frac{d}{dk}\tilde{\chi}^*(k) + \sigma^b_{Mm} \Phi^b_m \, , \text{ where}$$

$$\Phi^b_m = \int \frac{d^3p}{(2\pi)^3} \tilde{F}_m(\vec{p})\, \hat{p}^b \frac{d}{dp}\tilde{\chi}^*(p) \tag{8}$$

$$\tilde{F}_{m'}(\vec{q}) = \frac{-2m\lambda^2}{E-E_0} \frac{\frac{d}{dq}\tilde{\chi}(q)}{q^2-k^2} \hat{q}^a \sigma^a_{m'M} \sigma^b_{Mm} \left( \hat{z}^b \xi_m \frac{d}{dk}\tilde{\chi}^*(k) + \Phi^b_m \right)$$

$$= \frac{-2m\lambda^2}{E-E_0} \frac{\frac{d}{dq}\tilde{\chi}(q)}{q^2-k^2} \hat{q}^a (\sigma^a\sigma^b)_{m'm} \left( \hat{z}^b \xi_m \frac{d}{dk}\tilde{\chi}^*(k) + \Phi^b_m \right) \tag{9}$$

(9) → (8),

$$\Phi^b_{m'} = \frac{-2m\lambda^2}{E-E_0} \int \frac{d^3p}{(2\pi)^3} \frac{\hat{p}^b}{p^2-k^2} \left|\frac{d}{dp}\tilde{\chi}(p)\right|^2 \hat{p}^a (\sigma^a\sigma^b)_{m'm} \left[ \hat{z}^b \xi_m \frac{d}{dk}\tilde{\chi}^*(k) + \Phi^b_m \right]$$

$$= \frac{-2m\lambda^2}{E-E_0} X_{m'm}(k) \left[ \hat{z}^b \xi_m \frac{d}{dk}\tilde{\chi}^*(k) + \Phi^b_m \right], \tag{10}$$

where $\Sigma_{m'm}(k) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{p^2-k^2} \left|\frac{d}{dp}\tilde{\chi}(p)\right|^2 \hat{p}^a \hat{p}^b (\sigma^a\sigma^b)_{m'm}$

$$\hat{p}^a \hat{p}^b (\sigma^a \sigma^b)_{mm'} = \frac{1}{2}\left(\hat{p}^a\hat{p}^b\sigma^a\sigma^b + \hat{p}^b\hat{p}^a\sigma^b\sigma^a\right)_{m'm}$$

$$= \frac{1}{2}\hat{p}^a\hat{p}^b\{\sigma^a, \sigma^b\}_{m'm}$$

$$= \frac{1}{2}\hat{p}^a\hat{p}^b(2\delta^{ab}\delta_{m'm'})$$

$$= \hat{p}\cdot\hat{p}\,\delta_{m'm}$$

$$= \delta_{m'm}$$

$\Rightarrow \Sigma_{m'm}(k) = \Sigma\,\delta_{m'm}$, where

$$\Sigma(k) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{p^2-k^2} \left|\frac{d}{dp}\tilde{\chi}(p)\right|^2 \quad \text{as in prob. (a)}$$

$$\boxed{\Sigma_{m'm}(k) = \Sigma(k)\,\delta_{m'm}} \qquad (11)$$

(11) $\to$ (10), $\left(1 + \frac{2m\alpha^2}{E-E_0}\Sigma\right)\delta_{m'm}\,\Phi^b_m = \frac{-2m\alpha^2}{E-E_0}\Sigma\,\delta_{m'm}\,\xi_m\,\hat{z}^b\,\frac{d}{dk}\tilde{\chi}^*(k)$

$$\boxed{\Phi^b_m = -2m\alpha^2\,\Sigma(k)\,\xi_m\,\hat{z}^b\,\frac{d}{dk}\tilde{\chi}^*(k)\left(\frac{1}{E-E_0+2m\alpha^2\Sigma(k)}\right)} \qquad (12)$$

(12) $\rightarrow$ (9),

$$\tilde{F}_{m'}(\vec{q}) = \frac{-2md^2}{\cancel{E-E_0}} \frac{d}{dq}\tilde{\chi}(q) \frac{1}{q^2-k^2} \hat{q}^a (\sigma^a\sigma^b)_{m'm} \hat{z}^b \xi_m \frac{d}{dk}\tilde{\chi}^*(k)$$

$$\times \left( \frac{\cancel{E-E_0}}{E-E_0+2md^2\Sigma(k)} \right) \qquad (13)$$

$$\tilde{F}_{m'}(\vec{q}) = -2md^2 (\sigma^a\sigma^b)_{m'm} \hat{z}^b \xi_m \frac{d}{dk}\tilde{\chi}^*(k) \frac{1}{E-E_0+2md^2\Sigma(k)}$$

$$\times \frac{d}{dq}\tilde{\chi}(q) \frac{1}{q^2-k^2} \hat{q}^a$$

$$\Rightarrow F_{m'}(\vec{r}) = -2md^2 \frac{\frac{d}{dk}\tilde{\chi}^*(k)}{E-E_0+2md\Sigma(k)} (\sigma^a\sigma^b)_{m'm} \hat{z}^b \xi_m \int \frac{d^3q}{(2\pi)^3} \frac{e^{i\vec{q}\cdot\vec{r}}}{q^2-k^2} \hat{q}^a \frac{d}{dq}\tilde{\chi}(q)$$

$$\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{1}{q^2-k^2} \hat{q}^a \frac{d}{dq}\tilde{\chi}(q) = \frac{e^{ikr}}{4\pi r} \hat{r}^a \frac{d}{dk}\tilde{\chi}(k)$$ as is

found in prob (a), $kr \gg 1$

$$F_{m'}(\vec{r}) = \frac{e^{ikr}}{r} \left\{ \frac{-md^2}{2\pi} \frac{1}{(E-E_0+2md\Sigma(k))} \left|\frac{d}{dk}\tilde{\chi}(k)\right|^2 (\sigma^a\sigma^b)_{m'm} \xi_m \hat{z}^b \hat{r}^a \right\}$$

$$\sigma^a\sigma^b \hat{z}^b \hat{r}^a = (\hat{r}^a\sigma^a)(\hat{r}^b\sigma^b), \quad \hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$$= \left[ \sin\theta\cos\phi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sin\theta\sin\phi \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \cos\theta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right]$$

$$\times \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta(\cos\phi - i\sin\phi) \\ \sin\theta(\cos\phi + i\sin\phi) & -\cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta\, e^{-\phi} \\ \sin\theta\, e^{\phi} & \cos\theta \end{pmatrix}$$

$$F_{m'}(\vec{r}) = \frac{e^{ikr}}{r}\left( -\frac{m\lambda^2}{2\pi}\, \frac{1}{E-E_0+2m\lambda\Sigma} \left|\frac{d}{dk}\tilde{\chi}(k)\right|^2 \right) \begin{pmatrix} \cos\theta & -\sin\theta\, e^{-\phi} \\ \sin\theta\, e^{\phi} & \cos\theta \end{pmatrix}_{m'm} \xi_m$$

$$(f_k)_{m'} = -\frac{m\lambda^2}{2\pi}\, \frac{1}{E-E_0+2m\lambda^2\Sigma} \left|\frac{d}{dk}\tilde{\chi}(k)\right|^2 \begin{pmatrix} \cos\theta & -\sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & \cos\theta \end{pmatrix}_{m'm} \xi_m$$

$$\frac{1}{E-E_0+2m\lambda\Sigma} = \frac{1}{E-E_0+2m\lambda^2\Sigma(\lambda)+2m\lambda i H(k)} = \frac{1}{E-E^*+i\Gamma/2}$$

$$\Gamma/2 = 2m\lambda^2 \frac{\pi}{2\pi^2}\, \frac{k}{2} \left|\frac{d}{dk}\tilde{\chi}(k)\right|^2$$

$$= \frac{m\lambda^2}{2\pi} k \left|\frac{d}{dk}\tilde{\chi}(k)\right|^2$$

$$\Rightarrow (f_k)_{m'} = -\frac{1}{k}\, \frac{\Gamma/2}{E-E^*+i\Gamma/2} \begin{pmatrix} \cos\theta & -\sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & \cos\theta \end{pmatrix}_{m'm} \xi_m$$

$$= \frac{1}{k} e^{i\delta_1}\sin\delta_1 \begin{pmatrix} \cos\theta & -\sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & \cos\theta \end{pmatrix}_{m'm} \xi_m$$

If the incoming wave was in $m = +\frac{1}{2}$ state, $\Rightarrow \xi_m = \begin{pmatrix}1\\0\end{pmatrix}_m$,

then $$(f_k)_{m'} = \frac{1}{k} e^{i\delta_1} \sin\delta_1 \begin{pmatrix} \cos\theta & -\sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & \cos\theta \end{pmatrix} \begin{pmatrix}1\\0\end{pmatrix}_{m'}$$

$$= \frac{1}{k} e^{i\delta_1} \sin\delta_1 \begin{pmatrix} \cos\theta \\ \sin\theta\, e^{i\phi} \end{pmatrix}_{m'}$$

On the other hand, if the incoming wave was an $m = -\frac{1}{2}$ state, $\xi_m = \begin{pmatrix}0\\1\end{pmatrix}_m$, and

$$(f_k)_{m'} = \frac{1}{k} e^{i\delta_1} \sin\delta_1 \begin{pmatrix} -\sin\theta\, e^{-i\phi} \\ \cos\theta \end{pmatrix}_{m'}$$

To make sense of this result, let's write the general equation for $f_k$ in terms of spherical harmonics, instead of Legendre Polynomials,

$$(f_k)_l = \frac{1}{k} e^{i\delta_l} \sin\delta_l\, (2l+1) P_l(\cos\theta)$$

$$= \frac{1}{k} e^{i\delta_l} \sin\delta_l \sqrt{4\pi(2l+1)}\; Y_l^0(\theta),$$

Given an incoming wave in the state $\xi_+ e^{ikz}$, we claim that the probability amplitude for it to scatter into a solid angle $\Omega$ is given by the sum of the product of the amplitude to enter into the added state $|J, M\rangle$ and the amplitude to exit this state while conserving angular momentum. i.e.,

$$(f_k^{+})_m = \sqrt{4\pi(2l+1)}\ \frac{1}{k} e^{i\delta_l} \sin\delta_l \ \langle l=1, m_l=0; S=\tfrac{1}{2}, m_s=\tfrac{1}{2} \mid J=\tfrac{1}{2}, M=\tfrac{1}{2}\rangle$$

$$\times \sum_{m_l', m_s'} \langle J=\tfrac{1}{2}, M=\tfrac{1}{2} \mid l=1, m_l'; S=\tfrac{1}{2}, m_s'\rangle Y_{1m_l'}(\Omega)(\xi_{m_s'})_m$$

So,

$$(f_k^{+})_m = \sqrt{4\pi\times 3}\ \frac{1}{k} e^{i\delta_1} \sin\delta_1 \left(-\sqrt{\tfrac{1}{3}}\right)$$

$$\times \Big\{ \langle J=\tfrac{1}{2}, M=\tfrac{1}{2} \mid l=1, 0; S=\tfrac{1}{2}, \tfrac{1}{2}\rangle Y_{10}(\Omega)(\xi_+)_m$$

$$+ \langle J=\tfrac{1}{2}, M=\tfrac{1}{2} \mid l=1, 1; S=\tfrac{1}{2}, -\tfrac{1}{2}\rangle Y_{11}(\Omega)(\xi_-)_m \Big\}$$

$$= \sqrt{12\pi}\ \frac{1}{k} e^{i\delta_1} \sin\delta_1 \left(-\sqrt{\tfrac{1}{3}}\right)$$

$$\times \left\{ \left(-\sqrt{\tfrac{1}{3}}\right) \frac{1}{2}\sqrt{\frac{3}{\pi}} \cos\theta \begin{pmatrix}1\\0\end{pmatrix}_m + \sqrt{\tfrac{2}{3}}\left(-\tfrac{1}{2}\right)\sqrt{\frac{3}{2\pi}} \sin\theta\, e^{i\varphi} \begin{pmatrix}0\\1\end{pmatrix}_m \right\}$$

$$\equiv \frac{1}{k} e^{i\delta_l} \sin\delta_l \begin{pmatrix}\cos\theta\\ \sin\theta\, e^{i\varphi}\end{pmatrix}_m$$

Note that $|\langle J=\tfrac{1}{2}, M=\tfrac{1}{2} \mid l=1, m=0, S=\tfrac{1}{2}, m_s=\tfrac{1}{2}\rangle|^2 = \left(-\tfrac{1}{\sqrt{3}}\right)^2 = \tfrac{1}{3}$. This factor cancels the expected factor of $(2l+1)=3$ for p-wave scattering, hence the $\frac{1}{k}$ instead of $\frac{3}{k}$!

If the incoming wave is $S_z = -\frac{1}{2}$ state, we get,

$$(f_k^{-})_m = \sqrt{4\pi(3)}\ \frac{1}{k} e^{i\delta_1} \sin\delta_1 \ \langle l=1, m_l=0; S=\tfrac{1}{2}, m_s=-\tfrac{1}{2} \mid J=\tfrac{1}{2}, M=-\tfrac{1}{2}\rangle$$

$$\times \sum_{m_l', m_s'} \langle J=\tfrac{1}{2}, M=-\tfrac{1}{2} \mid l=1, m_l'; S=\tfrac{1}{2}, m_s'\rangle Y_{1m_l'}(\Omega)(\xi_{m_s'})_m$$

$$= \sqrt{12\pi}\ \frac{1}{k} e^{i\delta_1} \sin\delta_1 \left(\frac{1}{\sqrt{3}}\right) \Big\{ \langle \tfrac{1}{2}, -\tfrac{1}{2} \mid 1, 0; \tfrac{1}{2}, -\tfrac{1}{2}\rangle Y_{10}(\xi_-)_m$$

$$+ \langle \tfrac{1}{2}, -\tfrac{1}{2} \mid 1, -1; \tfrac{1}{2}, \tfrac{1}{2}\rangle Y_{1-1}(\xi_+)_m \Big\}$$

$$= \sqrt{12\pi}\ \frac{1}{k} e^{i\delta_1} \sin\delta_1 \left(\frac{1}{\sqrt{3}}\right) \left\{ \frac{1}{\sqrt{3}} \left(\frac{1}{2}\right) \sqrt{\frac{3}{\pi}} \cos\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix}_m \right.$$

$$\left. + \left(-\sqrt{\frac{2}{3}}\right) \left(\frac{1}{2}\right) \sqrt{\frac{3}{2\pi}} \sin\theta\, e^{-i\varphi} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_m \right\}$$

$$= \frac{1}{k} e^{i\delta_1} \sin\delta_1 \begin{pmatrix} -\sin\theta\, e^{-i\varphi} \\ \cos\theta \end{pmatrix}_m$$

The following generalization is well waranted;
If the added states are $|J, M\rangle$ multiplets, the scattering amplitude for an incoming partial wave in the $l, m_l$ orbital, and $s, m_s$ spin state to scatter into the solid angle $\Omega$ in the $s', m_s'$ spin state is given by

$$f_k^{s', m_s'} = \frac{1}{k} e^{i\delta_l} \sin\delta_l \sqrt{4\pi(2l+1)}$$

$$\times \sum_M \sum_{l', m_l'} \langle l, m_l; s, m_s | J M \rangle \langle J M | l', m_l'; s', m_s' \rangle Y_{l'}^{m_l'} \xi_{m_s'}$$

For example, if the incoming wave is $s=0$ (scalar), and the added states are $J=1$ multiplets, we should recover the result for problem (a), And we do!

$$f_k^{l m_l} = \frac{1}{k} e^{i\delta_1} \sin\delta_1 \sqrt{4\pi(3)}\ \langle 1,0;0,0|1,0\rangle\langle 1,0|1,0,0,0\rangle\ Y_1^0 \ (1)$$

$$= \frac{1}{k} e^{i\delta_1} \sin\delta_1 \sqrt{12\pi}\ Y_1^0 \quad = \frac{1}{k} e^{i\delta_1} \sin\delta_1 \,(3)\, P_1(\cos\theta)$$